Statistical Properties of the Pseudo Wigner-Ville Representation of Normal Random Processes

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Abstract

Cumulants of discrete time pseudo Wigner-Ville (PDWV) representation of a normal process are derived in both the real and complex cases. The PDWV follows a generalised Chi-square law. If the input process is white, all the odd cumulants happen to be independent of the frequency.

1 Introduction

In an increasing number of works dealing with decisions in the time frequency plane based on the Wigner-Ville representation [2][5], the statistical properties [8][1][7] of the two dimensional associated process plays a crucial role to elaborate Bayesian [3] detection-estimation procedures or even, at least, to choose a threshold. This short paper aims to give exact (non asymptotic) statistical properties of the discrete time pseudo Wigner-Ville representation (PDWV) of a normal random process. In section 2, the PDWV is written as a weighted summation of uncorrelated time-frequency atoms, for any second order process. We show in section 3 that the PDWV of a normal process follows a generalised Chi-square law. Finally, we give in section 4 the exact form of any cumulant of the PDWV of a white normal process. The main result stems from the fact that, under these latter assumptions, the odd cumulants do not depend upon the frequency. We extend the results to the complex case in section 5.

2 Algebraic form of the Wigner-Ville representation

Let $x[n]$ be a real valued random process with zero mean and nonstationary at any order $q \geq 2$. Its continuous (in frequency) Wigner-Ville representation is well known as [4]

$$W[n, \nu] = 2 \sum_{k=-N}^{N} x[n + k] x[n - k] e^{-jk\pi\nu}$$  \hspace{1cm} (1)
Because $W[n, \nu]$ is real, (1) is equivalent to

$$W[n, \nu] = 2 \sum_{k=-N}^{N} x[n + k] x[n - k] \cos(4\pi\nu k) \tag{2}$$

In the whole paper, we introduce the following notations:

$$X[n] = x[n]x[n]^T \tag{3}$$

where

$$x[n] = [x[n - N] \ldots x[n - 1], x[n], x[n + 1] \ldots x[n + N]]^T \tag{4}$$

$$[H(\nu)]_{ij} = 2 \delta[i + j - (2N + 2)] \cos(4\pi(N + 1 - i)\nu) \tag{5}$$

$1 \leq i, j \leq 2N + 1$

$\delta[n]$ is the Kronecker sequence, $\Gamma[n]$ denotes the covariance matrix of $x[n]$ and $C[n]$ is its square root

$$\Gamma[n] = E[X[n]] = C[n]C[n] \tag{6}$$

Introducing $y[n]$ the second order whitened observational vector:

$$y[n] = C^{-1/2}[n] x[n] \tag{7}$$

we consider finally the matrix:

$$A[n, \nu] = C[n] H(\nu) C[n] \tag{8}$$

which is supposed to have the spectral form of rank $p \leq 2N + 1$

$$A[n, \nu] = \sum_{k=1}^{p} \lambda_k[n, \nu] v_k[n, \nu] v_k^T[n, \nu] \tag{9}$$

In keeping with (2)-(9), the Wigner-Ville representation takes the two following algebraic structure

$$W[n, \nu] = Tr(H(\nu)X[n]) \tag{10}$$

$$W[n, \nu] = \sum_{k=1}^{p} \lambda_k[n, \nu] w_k[n, \nu] \tag{11}$$

where $w_k[n, \nu] = v_k^T[n, \nu] y[n]$ $k \in \{1 \ldots p\}$

are uncorrelated, zero mean, unit variance random variables.

The formulation (10) has two main advantages: (i) it presents the Wigner-Ville representation of $x[n]$ as a scalar product of two symmetrical matrices, what will help us in the determination of the statistical properties of $W[n, \nu]$: one stands for a frequency operator, the other for time bilinear observations and (ii) rewrites the Wigner distribution as a summation of weighted uncorrelated time-frequency atoms.

The expression (11) will be used in the sequel of this study to determine the statistical properties of the Wigner-Ville representation of some particular processes.

3 Distribution of cumulants under normality

If $x[n]$ is distributed as normal, $W[n, \nu]$ has the characteristic function

$$\phi_{W[n, \nu]}(z) = \prod_{k=1}^{p} \frac{1}{\sqrt{1 - 2iz\lambda_k[n, \nu]}} \tag{12}$$
In the general case, the variables \( w_k[n, \nu] \) for \( k \neq k' \) are zero mean, unit variance and uncorrelated because of the orthogonality of the eigenvectors \( \Phi_k[n, \nu] \) and the whiteness of the normalised components of \( y[n] \).

When \( x[n] \) is normally distributed, so is \( y[n] \), the \( w_k[n, \nu] \) are moreover jointly Gaussian. Their decorrelation becomes a statistical independence. The distribution of \( W[n, \nu] \) (11) is consequently a generalised Chi\(^2\) with \( p \) degrees of freedom, which characteristic function is known as (12) [6]. From the expansion of the logarithm of (12) and the definition of the cumulants, we deduce two equivalent explicit forms of the order \( q \) cumulant:

\[
\kappa_q[n, \nu] = 2^{q-1}(q-1)! \sum_{k=1}^{p} \lambda_k^n[n, \nu] \tag{13}
\]

where the \( \lambda_k[n, \nu] \) are the eigenvalues of \( A[n, \nu] \).

\[
\kappa_q[n, \nu] = 2^{q-1}(q-1)! Tr(A^q[n, \nu]) = 2^{q-1}(q-1)! Tr(\Gamma^q[n]H^q(\nu)) \tag{14}
\]

4 Cumulant expressions under white normality

Let \( x[n] \) be a nonstationary white normal process with zero mean and variance \( \sigma^2[n] \). The cumulants of \( W[n, \nu] \) verify

- the odd order cumulants are \textit{independent of the frequency variable} with value

\[
\kappa_{2q+1}[n, \nu] = 2^{q+1}(2q)! \sigma^{4q+2}[n] \quad \forall q \geq 0 \tag{15}
\]

- the even order cumulants depend on the two variables \( n \) and \( \nu \) and have the form

\[
\kappa_{2q}[n, \nu] = 2^{q-1}(2q-1)! \sum_{i=1}^{2N+1} \cos^q(4\pi(N + 1 - i)\nu) \sigma^{2n}[n + N + 1 - i] \sigma^{2n}[n - N - 1 + i] \tag{16}
\]

When the process \( x[n] \) is white and normally distributed, \( \Gamma[n] \) and then \( C[n] \) are diagonal matrices. We can remark that although \( \Gamma[n] \) is diagonal, it is not proportional to the identity because of the nonstationarity of \( x[n] \).

In this case, the matrix \( A[n, \nu] \) has a special structure: it is “anti-diagonal” (that means with nonzero components only on the nonprincipal diagonal) and moreover symmetrical

\[
[A[n, \nu]]_{ij} = 2 \delta[i + j - (2N + 2)] \cos(4\pi(N + 1 - i)\nu) \sigma[n + N + 1 - i] \sigma[n - N - 1 + i] \tag{17}
\]

for all \( 1 \leq i, j \leq 2N + 1 \)

- for odd order cumulants, \( A^q[n, \nu] \) is anti-diagonal and its trace has only one nonzero value, what leads to the expression (15) and a really surprising result: the odd order cumulants of \( W[n, \nu] \) in the white Gaussian case are \textit{independent of the frequency variable}.

We obtain for instance the classical expression of the mean of the Wigner-Ville representation

\[
\kappa_1[n, \nu] = 2 \sigma^2[n] \]

- for even orders, \( A^q[n, \nu] \) is diagonal and the cumulants are obtained with the help of (17) and this time depend on the two variables \( n \) and \( \nu \). Their calculation requires the knowledge of the variance function in time.

It is worth mentioning that in the case of white normality, the eigenvectors \( v_k \) of \( A[n, \nu] \) do not depend upon time \( n \) and frequency \( \nu \) since the specific structure of \( A[n, \nu] \) leads to three kinds of eigenvectors: \( v_0 = [1 \ 0 \ 0]^T \), \( v_k = [0 \ 1 \ 0]^T \) and \( v_k = [0 \ 0 \ 1]^T \). The eigenvalues are equal to \( \cos(4\pi(N + 1 - i)\nu) \sigma[n + N + 1 - i] \sigma(n - N - 1 + i) \) for \( 1 \leq i \leq N \), their opposite and 1. In the case of coloured normality, there is no reason why the eigenvectors should not depend upon time and frequency.
5 Complex case

The properties presented in the preceding sections in the real case extend naturally when considering the pseudo Wigner-Ville representation of a complex signal $z[n]$

$$W[n, \nu] = 2 \sum_{k=-N}^{N} z[n+k] z^*[n-k] e^{-j\pi k}$$

This quantity can be expressed, as in the real case, in the following compact form

$$W[n, \nu] = Tr\{E(\nu)Z[n]\}$$

where $Z[n]$ has the same structure as (3) and $E(\nu)$ is the complex version of (5) replacing $\cos(.)$ by $\exp(-j\cdot)$. Although the signal $z[n]$ is now complex valued, it’s pseudo Wigner-Ville representation is still real. We can therefore rewrite (19) in terms of the trace of real valued matrices. With the notations $z[n] = x[n] + jy[n]$ and $E(\nu) = A(\nu) + jB(\nu)$, we have

$$W[n, \nu] = Tr\{\tilde{E}(\nu)\tilde{Z}[n]\}$$

where

$$\tilde{Z}[n] = [\bar{x}^T[n] \bar{y}^T[n]]^T \quad \tilde{E}(\nu) = \begin{pmatrix} \tilde{A}(\nu) & \tilde{B}(\nu) \\ -\tilde{B}(\nu) & \tilde{A}(\nu) \end{pmatrix}$$

Therefore, introducing similar matrices as in the above sections, it turns out that the cumulants of order $q$ of (18) in case of normality are equal to

$$\tilde{\kappa}_q[n, \nu] = 2^{q-1}(q-1)! Tr(\tilde{\Gamma}^q[n]\tilde{E}^q(\nu))$$

where

$$\tilde{\Gamma}^q[n] = E\left[\tilde{Z}[n]\tilde{Z}^T[n]\right]$$

6 Simulations

In this section, we will support the theoretical results of the preceding section, and especially the results (15)-(16) by Monte Carlo simulations. In a first case, 500 samples of size $N=5000$ of Gaussian white noise have been generated. The corresponding Wigner-Ville representations have been derived (using 127 estimation points for each time and frequency) and their first four cumulants empirically estimated. Only 33 frequency points have been taken into account between the normalised frequencies 0 and 0.5. The normalised mean and variance of the estimation residuals have been tabulated (table 1) and we have drawn on figure 1 the shapes for the fourth order cumulant - where the dashed line represents the theoretical value obtained from (16).

Because the Wigner-Ville representation is dedicated to nonstationary studies, we have considered 500 samples of size $N=5000$ of Gaussian process, centered, but with variance linearly dependent of the time variable $n$. The cumulants of the Wigner-Ville form of $x[n]$ are now depending on two variables and have been estimated with adaptative cumulants estimators. We have thus drawn only the mean and the variance (figure 2 and 3) with no comparison with the theoretical results and reported the mean and variance of the residuals in table 1. We remark, as expected, that the mean is independent of the frequency variable, property that vanishes for the variance.

7 Conclusion and future work

According to different assumptions regarding the normality of a nonstationary real valued process, exact (non asymptotic) statistical properties of its PDWV distribution has been derived. They may lead to elaborate Bayesian structure of receivers in the relevant time-frequency plane. Future work will present the usefulness of the results pointed out in this paper for such a purpose.
Figure 1. 4th order cumulant of $W[n,v]$ when $x[n]$ is white normal

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_4$</th>
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<tr>
<td>stationary</td>
<td>$E \left[ \left( \frac{\mu_1 - \mu_2}{\mu_1} \right)^4 \right]$</td>
<td>$1.26 e^{-2}$</td>
<td>$1.5 e^{-3}$</td>
<td>0.77</td>
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<td>case</td>
<td>$E \left[ \left( \frac{\mu_1 - \mu_2}{\mu_1} \right)^2 \right]$</td>
<td>$4.57 e^{-4}$</td>
<td>$3.4 e^{-6}$</td>
<td>0.89</td>
</tr>
<tr>
<td>nonstationary</td>
<td>$E \left[ \left( \frac{\mu_1 - \mu_2}{\mu_1} \right)^4 \right]$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.81</td>
</tr>
<tr>
<td>case</td>
<td>$E \left[ \left( \frac{\mu_1 - \mu_2}{\mu_1} \right)^2 \right]$</td>
<td>$2.3 e^{-2}$</td>
<td>$6.6 e^{-2}$</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Table 1. $L_1$ and $L_2$ norms of the estimation residuals

References


Figure 2. *adaptative mean of $W[n, \nu]$ when $x[n]$ is normal and nonstationary*

Figure 3. *variance of $W[n, \nu]$ when $x[n]$ is normal and nonstationary*