1 Introduction - Context

Many papers have already been devoted to non-gaussian processes modeling [GUEGAN 1994] [MENDEL 1989] [NIKIAS 1985]. Besides the Middleton infinite series of weighted gaussian pdf [MIDDLETON 1977] which are very heavy to manipulate and to identify, three types of approaches might be considered.

- LF-NG Linear filtering of non-gaussian processes
- NLF-G Non linear filtering of gaussian processes
- NLF-NG Non linear filtering of non-gaussian processes

It has been noticed [DUVAUT 1993] that because of the limit central theorem effects, the impulse response of the filters in the LF-NG procedure must be very short if one wants to keep significant non gaussian behaviour. This paper gives a contribution to a particular class of NLF-G. The NLF is the simplest polynomial filter: a quadrature, and the gaussian input is a real or complex ARMA process. It yields a versatile class of non gaussian real or complex stochastic signals called QARMA processes, with an interesting compromise between a wide field of non gaussian characteristics and the possibility - in principle - to determine all the analytical forms of the polycorrelations and the calculation of non asymptotic variances of their estimators. It is the most simple Volterra filtering of a white gaussian input [NIKIAS 1993].

In section 2, three kinds of QARMA processes are defined, one real valued and two others QARMAN and QARMAS complex valued. Section 3 is devoted to the analytical forms of the second, third and fourth order correlations; their shapes are presented in the case of AR1 and AR2 filtering. It is worth mentioning that if the derivation of bi- and tricorrelation of real QARMA processes is straightforward, it becomes heavy for the complex QARMAN and QARMAS cases.

In section 4, the estimation aspects are addressed. In order to avoid the tedious calculation of more than 15000 terms - in the third order case - and more than 3000000 - in the fourth order case - to reach the variance of the estimators, we propose in section 4.2 an approach based on Hermite polynomials properties. A non asymptotic lower bound of the variance is obtained. The idea that consists in writing a non-linearity as a combination of Hermite polynomials has already been used by several authors to derive explicit statistical properties of non linearly filtered gaussian processes [BRILLINGER 1970]. As far as we know, it is the first use of this principle to derive a non asymptotic bound of the variance of a third order cumulant estimator.

2 Definition of QARMA signals

2.1 Real processes

The generation of QARMA processes requires two steps. First, we filter gaussian zero mean stationary noise by an ARMA filter and then we square the signal in order to exhibit non gaussian properties. Because we keep the gaussianity before the non-linearity via the ARMA filtering, we can characterize the signals only with the autocorrelation sequence. We can thereby give the theoretical forms of the polycorrelations of QAR processes regarding only the gaussian input autocorrelation. We have centered the non gaussian output by removing the mean, see figure 1.

2.2 Complex processes

The method of generation is almost the same, but two approaches are considered:

- Energy tests in non gaussian context, like in RADAR processing [BOUVET 1987], lead to square the norm of complex ARMA processes. We call QARMAN this type of signals (N for Norm). The output becomes real and therefore, we use real estimators.

- To keep complex signals, we have to square directly complex ARMA processes. Then the output has a non zero imaginary part and we must use complex
estimators of the polycorrelations. We call QARMAS this class of signals (...S for Squared).

Figure 1 sums up the three different classes we have generated.

In the whole sequel, we will consider only circular signals [PICINBONO 1995].

Circularity means that the following property is satisfied:

\[ z \text{ is circular } \iff \text{pdf}(\exp(i\theta) \cdot z) = \text{pdf}(z) \]  

The main consequence which follows is that a moment of \( n \) variables is zero if the number of conjugate terms and the number of non-conjugate terms are not equal.

\[ E \left[ \prod_{a_i-p} z^{a_i} \cdot \prod_{b_j-q} (z^*)^{b_j} \right] = 0 \quad \text{if} \quad p \neq q \]  

Therefore, in the complex case, \( y[n] \) (figure 1) is a gaussian circular process.

3 Statistical properties

3.1 Definition of polycorrelations

We present only the first four cumulants for complex inputs (3-4) as long as their definitions for real signals are simply a particular case of the complex ones.

If \( z_c[n] \) is a complex stationary and circular process with zero mean (the subscript 'c' is for centering), it’s second and fourth cumulant are defined as below:

\[ \gamma_{\text{comp}}^2[n] = E \left[ z_c[t] \cdot z_c^*[t+n] \right] \]  

\[ \gamma_{\text{comp}}^4[m,n,p] = \]  

\[ E \left[ z_c[t] \cdot z_c^*[t+m] \cdot z_c^*[t+n] \cdot z_c^*[t+p] \right] \]  

Because of the circularity constraint, the complex bicoherence is always zero, and in the sequel, \( \gamma_3[m,n] \) will designate the classical real bicoherence.

\[ \gamma_3[m,n] = E \left[ z_c[t] \cdot z_c^*[t+m] \cdot z_c^*[t+n] \right] \]  

Moreover, we have arbitrarily conjugated the last two factors of the fourth cumulant instead of any other combination. As a matter of fact, six different forms of the complex tricorrelation exist and the definition (4) is One tricorrelation [NIKIAS 1993].

3.2 Results for QARMA processes

\( z_c[n] \) is one of the three types of QARMA processes. Considering the circularity of \( y[n] \) in the complex case and the expansion of even order gaussian moments [SOULOUMIAC 1993], we can give the theoretical forms of the polycorrelations of \( z_c[n] \) regarding only \( \Gamma[n] \) the autocorrelation sequence of the \( y[n] \) ARMA sequence.

These results are collected on table 1.

3.3 Symmetry properties and shapes

- Bicorrelation case
  Besides the classical symmetries described by (6) [NIKIAS 1993], the bicoherence of a QARMA process is an even function in the bicorrelation plane \( \{m,n\} \) and therefore, is defined in an hexagonal symmetrical domain:

\[ \begin{align*} 
\gamma_3[m,n] &= \gamma_3[n,m] \\
\gamma_3[m,n] &= \gamma_3[m-n,-n] \\
\gamma_3[m,n] &= \gamma_3[n-m,-m] 
\end{align*} \]  

The three peculiar axes \( m = 0, n = 0 \) and \( m = n \) called principal axes carry pertinent characteristics (see figures 3-4); as a matter of fact, they exhibit the same variations than the correlation of the corresponding signal described on figure 2.

- Tricorrelation case
  The tricorrelation of a stationary signal is a function of three variables; and therefore, its representation takes place in a four dimension space. To ease its interpretation, we propose to draw the tricorrelations in particular subspaces (planes) we have called principal planes. To make a parallel with the bicoherence, we call the symmetry axes bi-principal axes. We have chosen the plane \( "p = m + n" \) because it yields a squared symmetry for QARMA processes.

In the complex case, we must take care of the conjugate terms in the definition (4). Indeed, the symmetries in the real case yield different forms of the complex tricorrelation and we have to introduce...
Table 1: Values of the first polycorrelations of QARMA processes

<table>
<thead>
<tr>
<th>QARMA</th>
<th>$2\Gamma^2[n]$</th>
<th>$8\Gamma[m] \Gamma[n] \Gamma[n-m]$</th>
<th>$16\Gamma(n)\Gamma(m)\Gamma(p-m)\Gamma(p-n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QARMAN</td>
<td>$</td>
<td>\Gamma[n]</td>
<td>^2$</td>
</tr>
<tr>
<td>QARMAS</td>
<td>$2\Gamma^2[n]$</td>
<td>$0$</td>
<td>$16\Gamma(n)\Gamma(p)\Gamma(n-m)\Gamma(p-n)$</td>
</tr>
</tbody>
</table>

Figure 2: Theoretical correlation of QAR1 and QAR2

Applying to the equations of Table 1 the classical forms for real and complex signals in appendix A - we can draw the theoretical shapes of the polycorrelations of the three QARMA classes. We have drawn the shapes only in the real case (figures 2-6) with the following values for the real AR1 and AR2 filtering

\[
\begin{align*}
QAR1 & \quad \Rightarrow \quad a = 0.9 \\
QAR2 & \quad \Rightarrow \quad r = 0.9 \text{ and } \theta = \pm 0.075
\end{align*}
\]
The shapes of the polycorrelations exhibit an exponential decrease on the (bi-)principal axes for a QAR1; coupled with oscillations for a QAR2. We can remark that as long as the poles’s moduli are equal, the polycorrelations of a QAR1 and a QAR21 are identical. In the case of the tricorrelation of a QARS, a principal direction has totally disappeared; and this is due to the signal’s circularity. Moreover, the complex tricorrelation is hermitian in the principal plane \( p = m + n \), see figures 5-6.

4 Estimation of QARMA attributes

4.1 Form of the estimators

In this paragraph, we briefly recall the classical forms of estimators, see [NIKIAS 1993]. We denote respectively \( \gamma_3 [m, n] \) and \( \gamma_4 [m, n] \) the unbiased empirical estimators of the bicorrelation and the tricorrelation in the principal planes. In order to reduce the calculation time of those estimators, we consider them only in the \( \Delta (P) \) domain defined in the \( \{m, n\} \) plane by : \( 0 \leq n \leq m \leq P - 1 \). We complete then the estimation using the symetries (6) and (8).

For the real case, we have:

\[
\gamma_3 [m, n] = \frac{1}{N-m} \sum_{t=0}^{N-m-1} z_v [t] z_c [t + m] z_r [t + n] \quad \forall (m, n) \in \Delta
\]

\[
\mu_4 [m, n] = \frac{1}{N-m-n} \sum_{t=0}^{N-m+n-1} z_v [t] z_c [t + m] z_r [t + n] z_r [t + m + n] \quad \forall (m, n) \in \Delta
\]

And then

\[
\gamma_4 [m, n] = \mu_4 [m, n] - \hat{\gamma}^2 [m] - \hat{\gamma}^2 [n] - \hat{\gamma} [m + n] \hat{\gamma} [n - m] \quad \forall (m, n) \in \Delta
\]

In the complex case, we must consider the empirical estimators of (4) and (7) to keep the estimation domain \( \Delta (P) \).

4.2 A theoretical non asymptotic lower bound of the variance of estimation

Asymptotic forms of estimators variance have already been derived by many authors [NIKIAS 1993]. We propose hereafter a non asymptotic bound, which makes no use of the limit central theorem.

Besides the non gaussian nature of the estimators, their accuracy has been pointed out only for the first two orders : the mean and the variance.

The two criteria are:

- Normalized bias :

\[
\frac{E [\hat{\gamma}_3 [m, n]]}{\gamma_3 [m, n]} - 1
\]  

(13)

- Normalized square root of the variance :

\[
\frac{ECT [\hat{\gamma}_4 [m, n]]}{\gamma_4 [m, n]}
\]  

(14)

With \( i \in \{3, 4\} \), which denotes the type of polycorrelation evaluated on the \( \Delta \) domain.

The theoretical calculus of the variance of estimation requires the derivation of 17,992 terms for the bicorrelation and 3,460,368 terms for the tricorrelation. The final value of these variances are intractable because of the very large number of terms. We propose here a less tedious approach based on Hermite polynomials : it is classical to consider a non-linearity as a combination of Hermite polynomials because of their properties (for more informations, refer to [ERDELYI 1953]), and we use this principle to reduce the calculation of the lower bound of the variance.

We use the orthogonality of the Hermite polynomials for the expectation weighted by a gaussian pdf and especially the relationship (15) where \( v \) and \( w \) are zero mean jointly gaussian variables with normalized correlation equal to \( \rho \). The proof of that extention of Mehler’s formula can be found in [GRANGER-NEWBOLD 1976].

\[
E [H_q (v) H_r (w)] = \rho^q \delta _{\mid q - r \mid}
\]  

(15)

We give the principle of the calculations only for \( \gamma_3 [0, 0] \).

First of all, we have to change the estimator (10) by replacing the powers of the \( z_v [t] \) process by the Hermite polynomials of the gaussian input \( y [t] \):

\[
\gamma_3 [0, 0] = \frac{1}{N} \sum_{t=0}^{N-1} z_v^2 [t]
\]
\[
\frac{1}{N} \sum_{t=0}^{N-1} (y^2[t] - \sigma^2)^3
\]

\[
= \frac{1}{N} \sum_{t=0}^{N-1} H_t(y[t]) + 12 H_4(y[t]) + 30 H_2(y[t]) + 8
\]  

(16)

Considering that \(y[t]\) is zero mean with variance equal to 1.

Making use of this new form and the equation (15), we can express \(E[\gamma_3^2[0,0]]\) and \(E^2[\gamma_3[0,0]]\) only with the correlation \(\rho\) of the \(y[t]\) process. We obtain

\[
\text{var}(\gamma_3[0,0]) = E[\gamma_3^2[0,0]] - E^2[\gamma_3[0,0]]
\]

\[
= \frac{1}{N^2} \sum_{t=0}^{N-1} \sum_{t'=0}^{N-1} 720 \rho^6_{t,-t} + 3456 \rho^4_{t,-t} + 1800 \rho^2_{t,-t}
\]  

(17)

where \(\rho_{t,-t} = E[y_t(t) y_{-t}(t')]\).

We can notice that the equation (17) collects all the terms of the correlation matrix of the process \(y[t]\). Converting the double sum into a single sum, (17) becomes:

\[
\text{var}(\gamma_3[0,0]) = \frac{5976}{N} + \frac{1}{N^2} \sum_{i=1}^{N-1} (N - i) \left[ 1440 \Gamma_y^0(i) + 6912 \Gamma_y^4(i) + 3600 \Gamma_y^2(i) \right]
\]  

(18)

The lower bound follows from equation (18) when the process \(y[t]\) is white; which means that all the \(\Gamma_y^i(i) \neq 0\) are lowered to zero.

Hence, a practical lower bound of the normalized square root of the variance (see (14)) is:

\[
\text{ECT} \, N \, [\gamma_3[0,0]] \geq \frac{\sqrt{5976}}{8\sqrt{N}} \approx \frac{9.7}{\sqrt{N}}
\]  

(19)

We stress the fact that the equation (19) gives an exact bound because it has not been derived using an asymptotic approximation just like the central limit theorem. The comparison with the same kind of lower bound for the second moment leads to the fact that in order to keep constant the accuracy of estimation of both second and third order moments of QARMA processes (real or not), the number of observation has to be approximately multiplied by 100 when switching from the second to the third order.

In the case of the tricorrelation, the estimator \(\gamma_3[0,0]\) can also be expressed with Hermite polynomials (20) ; but because we estimate the cumulant and not the fourth moment, the expression contains already products of two polynomials which will become products of four of them when calculating the variance of estimation. The calculus of the expectation requires therefore an extention of the formula (15) to the case of three and four variables. Slepian’s theorem [SLEPIAN 1972] allows this work but is not as easy as in the bicorrelation case and will be described in a future paper.

The accuracy’s simulations have been calculated on 16 points on the principal axe \(n = 0\) for the bicorrelation and the bi-principal axe \(n = 0\) \(p = m\) for the tricorrelation. The figures 7 - 8 show the norm of the criterions on the 16 considered points versus the number of estimation samples.

The bias is asymptotically zero because we have implemented the unbiased estimators of the polycorrelations. That is the reason why the standard deviation is so high; even for large number of samples. Figure 8 show that the non-asymptotic lower bound of the variance is relevant when we compare the curve of the variance of estimation for the QAR1 and the theoretical lower bound given by \(\frac{9.7}{\sqrt{N}}\) (in dashed line).

\[\gamma_3[0,0] = \frac{1}{N} \sum_{t=0}^{N-1} s_y^4[t] - \frac{3}{N^2} \left( \sum_{t=0}^{N-1} s_y^2[t] \right)^2\]

\[= \frac{1}{N} \sum_{t=0}^{N-1} H_3(t) + 24H_6(t) + 156H_4(t) + 272H_2(t) + 60\]

\[= \frac{3}{N^2} \left( \sum_{t=0}^{N-1} H_5(t) + 4H_3(t) + 2 \right)^2\]

(20)

5 Conclusion and future work

This paper gives a contribution to non gaussian processes modeling based on a squaring of an ARMA real or complex valued gaussian process. The relevant processes called QARMA processes exhibit a good compromise between a wide field of non-gaussian properties and the ease of analytical calculation of the polycorrelations. A new method based on Hermite polynomials has
be introduced to derive efficiently a non asymptotic bound of the variance of estimation of the polycorrelations. Several works have started to extend to other non-linearities the ideas contained in the paper.

A AR1 and AR2 correlations

A.1 Real processes

An ARp signal is generated by the equation:

$$y[n] = \sum_{i=1}^{p} a_i y[n-i] + x[n]$$  \hspace{1cm} (21)

Where $x[n]$ is a white process.

For an AR1, the correlation is:

$$\Gamma[n] = \frac{\sigma^2}{1 - \alpha^2} a[n] \quad \forall n \in \mathbb{Z}$$  \hspace{1cm} (22)

Where $\sigma$ is the standard deviation of the white gaussian input and $a$ the AR pole.

For an AR2 with two complex conjugated poles:

$$\Gamma[n] = \sigma^2 \beta r |n| \frac{\sin((n+1)\theta + \psi)}{\sin\theta} \quad \forall n \in \mathbb{Z}$$  \hspace{1cm} (23)

With

$$\beta = \frac{1}{(1-r^2)\sqrt{1 - 2r^2 \cos 2\theta + r^4}}$$

$$\tan\psi = \frac{r^2 \sin 2\theta}{1 - r^2 \cos 2\theta}$$

Where $r$ is the modulus of the poles and $\theta$ their normalized frequency.

A.2 Complex processes

In the complex case, and with respect to the form of the AR1 pole $a = |a| e^{2\pi \nu a}$, the correlation of an AR1 becomes:

$$\Gamma[n] = \frac{\sigma^2}{1 - |a|^2} (|a| e^{2\pi \mathrm{sgn}[n] \nu a}) |n| \quad \forall n \in \mathbb{Z}$$  \hspace{1cm} (24)

And in the case of an AR2, we can have the theoretical form of the correlation, but to reduce the calculations, we have considered two poles which have the same modulus $r$ and two different phases $\theta_1$ and $\theta_2$. We achieve therefrom:

$$\Gamma[n] = \sigma^2 \beta \left( r e^{i\pi \mathrm{sgn}(n)(n+1)(\theta_1 + \theta_2)} \right)^{|n|} \times \frac{\sin((n+1)\pi(\theta_1 - \theta_2) + \psi)}{\sin \pi(\theta_1 - \theta_2)} \quad \forall n \in \mathbb{Z}$$  \hspace{1cm} (25)

With

$$\beta = \frac{1}{(1-r^2)\sqrt{1 - 2r^2 \cos 2(\theta_1 - \theta_2) + r^4}}$$

$$\tan\psi = \frac{r^2 \sin 2(\theta_1 - \theta_2)}{1 - r^2 \cos 2(\theta_1 - \theta_2)}$$

References


