Abstract—For a \((\lambda(x), \rho(x))\) standard irregular LDPC code ensemble, the growth rate of the average weight distribution for small relative weight \(\omega\) is given by \(\log(\lambda'(0)\rho'(1))\omega + O(\omega^2)\) in the limit of code length \(n\). If \(\lambda'(0)\rho'(1) < 1\), there exist exponentially few code words of small linear weight, as \(n\) tends to infinity. It is known that the condition coincides with the stability condition of density evolution over the erasure channels with the erasure probability 1. In this paper, we show that this is also the case with multi-edge type LDPC (MET-LDPC) codes. MET-LDPC codes are generalized structured LDPC codes introduced by Richardson and Urbanke. The parameter corresponding \(\lambda'(0)\rho'(1)\) appearing in the conditions for MET-LDPC codes is given by the spectrum radius of the matrix defined by extended degree distributions.

I. INTRODUCTION

In 1963, Gallager invented low-density parity-check (LDPC) codes. Due to the sparseness of the code representation, LDPC codes are efficiently decoded by belief propagation (BP) decoders. By a powerful optimization using density evolution, invented by Richardson and Urbanke, messages of BP decoding can be statistically evaluated and optimized LDPC codes can approach very close to Shannon limit.

Recently, many structured LDPC codes have been proposed: accumulate repeat accumulate codes [18], irregular repeat accumulate codes [5], MacKay-Neal codes [8], protograph codes [12], raptor codes [4], low-density generator-matrix codes [7]. Above all, multi-edge type LDPC (MET-LDPC) codes [2] give us a framework which unifies all these codes.

Weight distributions of standard irregular LDPC codes are studied in [15], [17], [10], [16], [3] and structured codes are studied in [6], [14], [13]. Code Words of small weight mainly contributes decoding errors for high SNR regions. Constructing LDPC codes without small weight code words helps us to lower the error floors. In this paper, we present a simple expression of the number of code words for MET-LDPC codes. Moreover, we investigate the average number of code words and its asymptotic growth rate for small linear code word weight.

The rest of this paper is organized as follows. In Section II, we briefly review the definition of MET-LDPC codes. Section III derives the weight distributions of MET-LDPC codes. Section IV presents asymptotic analysis of the exponent \(\gamma(\omega)\) of the weight distributions for normalized weight \(\omega\). Section V concludes the paper.

II. MULTI-EDGE TYPE LDPC CODE

An MET-LDPC code ensemble is given in Fig. 1. The edges in the graphs can be divided into 5 types of edges. (Note that there are two types of edges in the third row of edges.) Furthermore variable and check nodes are classified into types according to the number of each edge-type they have. The types are called variable and check node-types, respectively. For instance, there are two types of check nodes in the second row of check nodes. Both node-types have two zig-zag edges from below. From above, one check node-type has one solid edges and two dashed edges, the other has two solid edges and one dashed edges. MET-LDPC codes are constructed by randomly connecting each types of edges between specified types of variable and check nodes.

The original definition of MET-LDPC [2] codes involves transmissions over \(n_t\) types of parallel channels. Since our interest in this paper is limited to weight distributions, we restrict ourselves to the transmissions over a single channel,
i.e. $n_{\ell} = 1$. There are 48 variable nodes in a graph in Fig. 1. We assume that 8 light blue variable nodes are punctured, i.e. these nodes are not transmitted over the channel. Therefore the length of the code turns out to be 40. Let $n_{\ell}$ be the number of edge-types. A node is said to be of node-type $d = (d_1, d_2, \ldots , d_\ell)$ if the node has $d_\ell$ edges of type $i$ ($i = 1, 2, \ldots , n_{\ell}$). Every variable node has its channel-type according to the channel for which the corresponding bit is sent. In this paper, punctured nodes are defined to have channel-type $b = (1, 0)$ and the others have $b = (0, 1)$. Numbering the edge-type from bottom left to top right as 1, 2, 3, 4, 5, we have 20, 12, 8 and 8 variable nodes of type (2, 0, 0, 0, 0), (0, 3, 0, 0, 0), (0, 0, 3, 0, 0) and (0, 0, 0, 3, 0), respectively. We have 16, 4 and 8 check nodes of type (2, 1, 0, 0), (2, 1, 2, 0, 0) and (0, 0, 0, 0, 1), respectively.

An MET-LDPC code ensemble is specified by a multivariate polynomial pair $\nu(r, x), \mu(x)$.

$$
\nu(r, x) := \sum_{d} \nu_{d} r_{d} x^{d}, \quad \mu(x) := \sum_{d} \mu_{d} x^{d}
$$

Then we define an equi-probable ensemble of graphs $G$ that satisfy the followings and denote by $C(n, \nu(r, x), \mu(x))$.

1) $G$ has $n$ variable nodes of channel-type (0, 1), i.e. $G$ has $n$ un-punctured variable nodes.

2) $G$ has $\nu_{d_{\ell}}$ variable nodes of channel-type $b$ and node-type $d$.

3) $G$ has $\mu_{d}$ check nodes of node-type $d$.

It is easy to see that the number of edges of edge-type $i$ connecting to variable and check nodes of node-type $d$ is given by $\sum_{d} d_{\nu_{d_{\ell}} d} = d_{\nu}(1, 0, d) + d_{\nu}(0, 1, d)$ and $\sum_{d} d_{\mu_{d}}$, respectively. It follows that the number of edges of edge-type $i$ connecting to variable nodes and check nodes are respectively as follows.

$$
n_{\nu_{i}}(1, 1) := n \frac{\partial}{\partial \nu_{i}} \nu(r, x) \bigg |_{r_{1}=1} = n \sum_{b_{d}} d_{\nu_{b_{d}, d}}
$$

$$
n_{\mu_{i}}(1) := n \frac{\partial}{\partial \mu_{i}} \mu(x) \bigg |_{x_{1}=1} = n \sum_{d} d_{\mu_{d}}
$$

They are constrained to be identical, i.e. $n_{\nu_{i}}(1, 1) = n_{\mu_{i}}(1)$. In this section, we see that the graph shown in Fig. 1 is an MET-LDPC code extracted from an ensemble $C(n = 40, \nu(r, x), \mu(x))$, where

$$
\nu(r, x) = 0.5r_{1} x_{1}^{2} + 0.3r_{2} x_{2}^{2} + 0.3r_{3} x_{3}^{2} + 0.2r_{0} x_{4} x_{2} + 0.2r_{1} x_{5}
$$

$$
\mu(x) = 0.4x_{1} x_{2} x_{3} + 0.1x_{2} x_{3} x_{2} x_{3} + 0.2 x_{3} x_{4} x_{5}
$$

III. WEIGHT DISTRIBUTION OF MULTI-EDGE TYPE CODES

The weight of code word is defined as the number of un-punctured variable nodes of value ‘1’. In this section, we derive average weight distribution of MET-LDPC code ensemble $C(n, \nu(r, x), \mu(x))$.

**Theorem 1:** The average number $A(\ell)$ of code words of weight $\ell$ for MET-LDPC code ensemble $C(n, \nu(r, x), \mu(x))$ is given as follows.

$$
A(\ell) = \sum_{e} \text{coef}((Q(l, s) P(u))^{n_{l}}, e, s, u^{e}) \prod_{i} \left( \frac{\mu_{i}(1)}{\nu_{i}(1)} \right),
$$

$$
Q(l, s) = \prod_{b, d} \left( 1 + e^{l_{b_{d}}} s^{d_{b_{d}}} \right)^{n_{b_{d}, a}},
$$

$$
P(u) = \prod_{d} \left( \frac{(1 + u)^{d_{i}} + (1 - u)^{d_{i}}}{2} \right)^{\mu_{d}},
$$

where, coef$(g(x), x^{d})$ is a coefficient of a term $x^{d} := \prod_{i} x_{i}^{d_{i}}$ of a multivariate polynomial $g(x)$.

**Proof:** An edge is said to be active, if the edge connects to a variable nodes of value ‘1’. We will count all code words of weight $\ell$ in all graphs in the ensemble for given $e_{i}$ active edge of edge-type $i$, and sum them up for all $e_{i}$. Counting code words involves three parts: and active edge constellation for check constraints, active edge constellation for code words of weight $\ell$, and edge permutations according to activeness and type of edges.

First, consider a check node $c$ of node-type $d$ with $e_{i}$ active edges of edge-type $i$ connecting to $c$ for $i = 1, 2, \ldots , n_{\ell}$. The check node $c$ is satisfied if the total number $\sum_{i} e_{i}$ of active edges is even. In this case, the number $a(e)$ of constellations of active edges to satisfy the check node $c$ is given by $\prod_{i} \left( \frac{d_{i}}{e_{i}} \right)$. And there are no constellation of active edges to satisfy $c$ if $\sum_{i} e_{i}$ is odd, i.e. $a(e) = 0$. It follows that the generating function $\sum_{e} a(e) u^{e} := \sum_{e} a(e) \prod_{i} \left( \frac{d_{i}}{e_{i}} \right)$ of the number $a(e)$ of constellations of active edges is given by

$$
f_{d}(u) := \sum_{e} \sum_{i \text{ even}} \frac{1}{2} \prod_{i} \left( \frac{d_{i}}{e_{i}} \right) u^{e} = \prod_{i}(1 + u)^{d_{i}} + \prod_{i}(1 - u)^{d_{i}}
$$

Next, consider all $n_{\mu}(1)$ check nodes and $e_{i}$ active edges of type $i$ for $i = 1, 2, \ldots , n_{\ell}$ are connecting to the check nodes. Since there are $n_{\mu_{d}}$ check nodes of type $d$, the number of constellations of active edges to satisfy all check nodes is given by

$$
\text{coef} \left( \prod_{d} f_{d}(u)^{n_{\mu_{d}}}, u^{e} \right).
$$

Secondly, consider a variable node $v$ of channel-type $b$ and node-type $d$. Assume $v$ has active $e_{i}$ edges of edge-type $i$ ($i = 1, 2, \ldots , n_{\ell}$). The weight of the variable node $v$ is defined to be 1 if $v$ is un-punctured and of value ‘0’ or punctured. The number $a(l, e)$ of right constellations of active edges is 1 if ($\ell = 1, e = d$) or ($\ell = 0, e = 0$), and 0 otherwise. Taking puncturing into consideration, we obtain the generating function of $a(l, e)$ as
follows.

\[ \sum_{i, e} a(\ell, e) i^\ell s^e = 1 + t^{e_1} s^{d} \]

Next, consider all \( n \) variable nodes. It is consequent that the number of possible active edge constellations for code words of weight \( \ell \) is given by

\[ \text{coef}(\prod_{b, d} (1 + t^{e_1} s^{d})^{n_{\beta_0} - d}, t^\ell s^{e}). \]  

(2)

In the third place, the number of possible ways of connecting edges between variable and check nodes so that \( e_i \) edges are of type \( i \) for \( i = 1, 2, \ldots, n_\ell \) is given by

\[ \prod_i e_i! \prod_i (n_{\mu_i}(1) - e_i)! \]  

(3)

Thus, multiplying (1), (2) and (3) divided by the total number of code in the ensemble, we obtain the average number of code words of weight \( \ell \) as follows

\[ A(\ell) = \sum_{\mathbf{e}} \frac{1}{e_i! \mu_i(1)!} \prod_i e_i! \prod_i (n_{\mu_i}(1) - e_i)! \times \text{coef}(\prod_{b, d} (1 + t^{e_1} s^{d})^{n_{\beta_0} - d}, t^\ell s^{e}) \times \text{coef}(\prod_{d} f_d(u)^{n_{\beta_0} - d}, u^e) \times \frac{\text{coef}((Q(t, s) P(u))^n, (t^\ell s^e u^\beta)^n)}{\prod_i (n_{\beta_0}(1)^n)} = \sum_{\mathbf{e}} \text{coef}((Q(t, s) P(u))^n, (t^\ell s^e u^\beta)^n). \]  

IV. ASYMPTOTIC ANALYSIS

LDPC codes are usually used with large code length, we are interested in asymptotic weight distributions in the limit of code length. First let us introduce the following lemma.

**Lemma 2 ([17], Theorem 2):** For a \( m \)-variable polynomial \( g(x_1, \ldots, x_m) \) with non-negative coefficients, it holds that

\[ \lim_{n \to \infty} \frac{1}{n} \log \text{coef}(g(x)^n, x^{\alpha}) = \inf_{x > 0} \log \frac{g(x)}{x^{\alpha}}, \]

where \( x > 0 \) means \( x_i > 0 \) for all \( i = 1, 2, \ldots, n_\ell \). The point \( x \) that takes \( \text{inf}_{x > 0} \) is given by a solution of the following equations:

\[ \frac{x_i}{g(x)} \frac{\partial g(x)}{\partial x_i} = a_i \quad (i = 1, 2, \ldots, m) \]

The possible ways of \( e \) in \( \sum_{\mathbf{e}} \) in Theorem 1 of the average weight distribution \( A(\ell) \) is upper-bounded by some polynomial of \( n \). Therefore the largest term alone contributes the exponent of \( A(\ell) \). Furthermore, using Lemma above, we can obtain the exponent \( \gamma(\omega) := \lim_{n \to \infty} \frac{1}{n} \log A(\omega n) \) of the average weight distribution \( A(\omega n) \) for weight \( \omega n \), where \( \omega \) is the normalized weight of code words.

\[ \lim_{n \to \infty} \frac{1}{n} \log \sum_{\beta_0} \text{coef}((Q(t, s) P(u))^n, (t^\ell s^e u^\beta)^n) \prod_{i=1}^{n_\ell} (n_{\beta_0}(1)^n) = \sup_{\beta \in \mathbb{B}(\omega)} \inf_{t, u, s} \left[ \log Q(s, t) + \log P(u) - \sum_{i=1}^{n_\ell} \beta_i \log(u_i) - \sum_{i=1}^{n_\ell} \beta_i \log(s_i) - \omega \log(t) - \sum_{i=1}^{n_\ell} \mu_i(1) h \left( \frac{\beta_i}{\mu_i(1)} \right) \right] =: \sup_{\beta \in \mathbb{B}(\omega)} \gamma(\beta) \]  

(4)

A point \( (u, s, t) \) that takes \( \inf_{t, u, s} \) is given as a solution of the following equations.

\[ \omega = \frac{\partial Q}{\partial t} = \sum_{b, d} \sum_{u, s} \frac{\partial Q}{\partial u} = \sum_{b, d} \sum_{u, s} \frac{\partial Q}{\partial s} = \sum_{b, d} \sum_{u, s} \frac{\partial Q}{\partial t} = \sum_{b, d} \sum_{u, s} \frac{\partial Q}{\partial s} \]  

(5)

A point \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n_\ell}) \) which gives \( \sup_{\beta \in \mathbb{B}(\omega)} \gamma(\beta) \) to satisfy the stationary condition

\[ \frac{\beta_i}{\mu_i(1) - \beta_i} = u_i s_i, \]  

(8)

Thus we obtain the following theorem.

**Theorem 3:** For a given MET-LDPC code ensemble \( \mathcal{C}(n, \nu(r, x), \mu(x)) \), the exponent of the number of code words of weight \( \omega n \) is given by

\[ \gamma(\omega) := \lim_{n \to \infty} \frac{1}{n} A(\omega n) = \max_{\beta \in \mathbb{B}(\omega)} \gamma(\beta), \]

where \( \mathbb{B}(\omega) \) is a set of \( \beta \) such that (5), (6), (7) and (8) hold.

The derivative of \( \gamma(\beta) \) in terms of \( \omega \) can be expressed in a simple expression.

**Lemma 4:** For \( \beta \) and \( t \) such that equations (5), (6) and (7) hold, we have the following.

\[ \frac{d}{d\omega} \gamma(\omega) = -\log(t(\omega)) \]

**Proof:** Let \( x' \) denote the deriviation of \( x \) with respect to \( \omega \). Differentiating \( \gamma(\omega) \) defined in (4), we have

\[ \frac{d}{d\omega} \gamma(\beta) = Q' + P' - \frac{t}{u} - \sum_{i=1}^{n_\ell} \log \left( \frac{\mu_i(1)}{\beta_i} \right) \frac{\beta_i}{\beta_i} \]

\[ - \log t - \sum_{i=1}^{n_\ell} (\beta_i' log u_i + \beta_i u_i'+ \beta_i' log s_i + \beta_i s_i') \]

where \( s \) is given by equations (5), (6) and (7). From (8), we see

\[ -\beta_i' \log u_i - \beta_i' \log s_i - \beta_i' \log \frac{\mu_i(1)}{\beta_i} = 0. \]
Combining (6) and $P' = \sum_i \frac{\partial P}{\partial u_i} u_i$, we have

$$\frac{P'}{P} - \sum_{i=1}^{n_t} \beta_i \frac{u_i}{u_i} = 0 \quad (9)$$

From (5), (7) and $Q' = \frac{\partial Q}{\partial t} t' + \sum_i \frac{\partial Q}{\partial s_i} s_i$, we have

$$\frac{Q'}{Q} - \frac{t'}{t} + \sum_i \beta_i \frac{s_i}{s_i} = 0$$

Thus we can conclude the proof since the remaining term in the right-hand side of (9) is $-\log t$.

A. Analysis of Small Weight Code Word

In this section we restrict ourselves to considering unpunctured MET-LDPC codes, i.e. $b = (0, 1)$ if $r_{b, d} \neq 0$. And for every edge-type $i$, we assume that there exists check node-type $d$ that $d_i \geq 2$. For the standard irregular LDPC codes [11] with a degree distribution pair $(\lambda(x), \rho(x))$, this assumption reduces to the condition that the maximum degree of check node is greater than 1, i.e. $\rho'(1) > 0$.

We investigate how the exponent of the average weight distribution behaves for code words of small weight, i.e. the normalized weight $\omega$ is small. From the linearity of MET-LDPC codes, $A(0) = 1$ and $\gamma(0) = 0$, then from (4) and Lemma 4, it follows that for $\omega \to 0$,

$$\gamma(\omega) = \gamma'(0) \omega + o(\omega) = \sup_{t \in T} -\log(t)\omega + o(\omega), \quad (10)$$

where $T$ is a set of $t$ such that (5), (6), (7) and (8) hold for $\omega \to 0$. From the assumption of non-puncturing and (5), for $\omega \to 0$, it holds that $ts^d \to 0$ for $d$ with $r_{b, d} \neq 0$. We see that, $\beta_i \to 0$ from (7) and the assumption of check node-types. Consequently, $u_i = 0$ from (6). Moreover, from (6) it follows that as $\omega \to 0$

$$\beta_i = \sum_d \mu_d u_i d_i ((d_i - 1) u_i + \sum_{j \neq i} d_i u_j) + o((\sum_i u_i)^2)$$

Substituting this to (8), we have

$$s_i = \frac{\mu_{i, i}(1)}{\mu_i(1)} u_i + \sum_{j \neq i} \frac{\mu_{i, j}(1)}{\mu_i(1)} u_j + o((\sum_i u_i)^2) \quad (11)$$

As $s \to 0$, from (7) we have

$$\beta_i = ts_i (\nu_{i, i}(1, 0) s_i + \sum_{j \neq i} \nu_{i, j}(1, 0) s_j) + o((\sum_i s_i)^2)$$

Substituting this to (8), we obtain the following.

$$u_i = t \left( \frac{\nu_{i, i}(1, 0)}{\nu_i(1, 1)} s_i + \sum_{j \neq i} \frac{\nu_{i, j}(1, 0)}{\nu_i(1, 1)} s_j \right) + o((\sum_i s_i)^2) \quad (12)$$

We can represent (11) and (12) by matrices as $s = Pu$ and $u = t\Lambda(1)s$, respectively, where

$$\Lambda_i(x) := \left. \frac{\partial \rho(x)}{\partial x} \right|_{x=0}, \quad P_{i,j} := \left. \frac{\partial \mu_i(x)}{\partial x} \right|_{x=1}.$$

In summary, we obtain

$$\frac{1}{t} u = \Lambda(1)P u + o((\sum_i u_i)). \quad (13)$$

This implies that $\frac{1}{t}$ is an eigenvalue of $\Lambda(1)P$. Therefore $\sup_{t \in T}$ of (10) is achieved by the largest eigenvalue $\frac{1}{t}$ of $\Lambda(1)P$. Then we have the following theorem.

**Theorem 5:** For an MET-LDPC code ensemble $C(n, \nu(r, x), \rho(x))$, the exponent $\gamma(\omega)$ of the average number of $A(\omega)$ code words of weight $\omega n$, in the limit of code length, is given by

$$\gamma(\omega) := \lim_{n \to \infty} \frac{1}{n} A(\omega n) = \log \left( \frac{1}{\Lambda} \right) \omega + O(\omega^2),$$

where $\Lambda$ is the largest eigenvalue of $\Lambda(1)P$. Furthermore, there exists $\delta > 0$ such that if $\frac{1}{t} < 1$, there are exponentially few code words of weight $\omega n$ for $\omega < \delta$.

For a standard irregular code ensemble with a given degree distribution pair $(\lambda(x), \rho(x))$, it can be seen that the same parameter $\lambda'(0)\rho'(1)$ and the condition reduces to $\lambda'(0)\rho'(1) < 1$, which coincides with a known result [15].

B. Relation with Stability Condition

We assume the transmission takes place over the BEC with the erasure probability $\epsilon$. We denote the average erasure probability of message sent from variable nodes to check nodes by $p^{(\ell)}$. From density evolution [9], $p^{(\ell)}$, in the limit of the code length, is given by

$$p^{(\ell)} = \epsilon (1 - \rho(1 - p^{(\ell-1)})), \quad p^{(0)} = \epsilon.$$

The following is known as the stability condition of density evolution. If $\epsilon \lambda'(0)\rho'(1) > 1$, there exists $\gamma > 0$ such that $\lim_{n \to \infty} p^{(\ell)} > \gamma$. Meanwhile, it is known that the exponent of the average number of code words of small linear weight $\omega n$ is given by

$$\lim_{n \to \infty} \frac{1}{n} \log A(\omega n) = \log(\lambda'(0)\rho'(1))\omega.$$

It can be seen that the same parameter $\lambda'(0)\rho'(1)$ appears in both the stability condition and the exponent of the number of code words. Does the correspondence holds also for the MET-LDPC code ensembles?

For the MET-LDPC code ensemble $C(n, \nu(r, x), \rho(x))$, let us assume the transmission over the BEC. Let $p^{(\ell)}_i$ denote the erasure probability of the message over edges of type $i$ from variable nodes to check nodes. From density evolution, it is known that $p^{(\ell)}_i$ is given by

$$p^{(\ell)} = \lambda((1, \epsilon), 1 - \rho(1 - p^{(\ell-1)})).$$
where $\lambda_i(r, x) = \frac{\nu_i(r, x)}{\nu_i(1)}$, $\rho_i(x) = \frac{\mu_i(x)}{\mu_i(1)}$ and $p_i^{(0)} = \epsilon$ for $i = 1, 2, \ldots, n$. The stability condition is given as follows. If the spectrum radius of $\Lambda(1, \epsilon)P$ is less than 1, there exists $\gamma > 0$ such that $\lim_{\ell \to \infty} \sum_i p_i^{(\ell)} > \gamma$. Since $\Lambda(1, \epsilon)P$ is a non-negative matrix, the spectrum radius of $\Lambda(1, \epsilon)P$ is an eigenvalue of $\Lambda(1, \epsilon)P$. It follows $\Lambda(1, 1)P$ coincides with the parameter which appears in Theorem 5.

V. CONCLUSION

We present a simple expression of the weight distributions of MET-LDPC code ensembles which gives us a general framework of LDPC codes. We showed that the correspondence between the exponent of the weight distributions and the stability condition is also the case with the MET-LDPC codes.

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