LOW-COMPLEXITY ITERATIVE RECONSTRUCTION
ALGORITHMS IN COMPRESSED SENSING

Graduate Student: Ludovic Danjean
Advisors: Bane Vasić, Michael W. Marcellin and David Declercq

Dept. of Electrical and Computer Eng. ETIS UMR 8051
University of Arizona ENSEA / Univ. Cergy-Pontoise / CNRS
Tucson, AZ 85721, USA F-95000 Cergy-Pontoise, France

E-mails: {danjean,vasic,marcellin}@ece.arizona.edu,
          {declercq}@ensea.fr

Abstract

In this paper we focus on two low-complexity iterative reconstruction algorithms in compressed sensing. These algorithms, called the approximate message-passing algorithm and the interval-passing algorithm, are suitable to recover sparse signals from a small set of measurements. Depending on the type of measurement matrix (sparse or random) used to acquire the samples of the signal, one or the other reconstruction algorithm can be used. We present the reconstruction results of these two reconstruction algorithms in terms of proportion of correct reconstructions in the noise free case. We also report in this paper possible practical applications of compressed sensing where the choice of the measurement matrix and the reconstruction algorithm are often governed by the constraint of the considered application.

1. INTRODUCTION

Compressed sensing [1] is a relatively new field of signal processing which concerns the recovery of a sparse high-dimensional signal from a small set of measurements. A $k$-sparse signal $x \in \mathbb{R}^n$, i.e. a signal $x$ with at most $k$ non-zero values, is observed indirectly through a shorter measurement vector $y \in \mathbb{R}^m$ that is obtained from the linear equations $y = Ax$ where $A$ is an $m \times n$ measurement matrix, with $m \ll n$. The task of compressed sensing is to recover $x$ from $y$. The first approach to solve the compressed sensing problem is to find a signal $x$ with the smallest $\ell_0$-norm satisfying $y = Ax$. The $\ell_0$-norm minimization of compressed sensing is NP-hard [1], [2]. For this reason, the $\ell_1$-norm minimization solution based on linear programming (LP) was introduced to reconstruct $x$. The LP technique [3] for the compressed sensing problem, called Basis Pursuit [4], [5], has remarkable performance, but its high complexity and running time makes it impractical in some applications which require fast reconstruction, or when the dimension of the signal is very large.

To tackle the issue of complexity, message-passing algorithms for compressed sensing have been proposed. These algorithms have their origins in channel coding. Sarvothan et al. [6] were among the first to introduce a reconstruction algorithm based on belief propagation. Another
application of belief propagation in compressed sensing was presented in [7] where two low-complexity algorithms are provided, list decoding and multiple-based belief propagation.

In this paper we focus on two message-passing reconstruction algorithms. The first one belongs to the class of iterative thresholding algorithm and is called the approximate message passing algorithm (AMPA) [8]. The AMPA exhibit excellent mean-square error reconstruction performance and can be analyzed theoretically using the so-called phase transition diagram. The second algorithm considered in this paper is a very simple message passing algorithm, which we refer to as the interval-passing algorithm (IPA) [9]. This simple iterative algorithm aims at reconstructing non-negative vectors. We have studied its reconstruction possibilities as well as comparing its reconstruction performance in [10].

In this paper, we consider \( k \)-sparse non-negative signals and we study the effect of the choice of measurement matrices. For a given \( n \) and \( m \) we consider either random Gaussian measurement matrices, or deterministic sparse non-negative measurement matrices using AMPA or IPA as a reconstruction algorithm. We then review some possible applications of compressed sensing. The main goal of this paper is to show how using sparse measurement matrices with appropriate low-complexity reconstruction methods affects the reconstruction performance.

The rest of the paper is organized as follows. Section II provides preliminaries on compressed sensing and a brief introduction to some iterative reconstruction algorithms. Section III presents the AMPA, while Section IV provides a detailed explanation of the IPA. Section V presents an overview of applications of compressed sensing. In Section VI, we provide simulation results exhibiting the IPA and the AMPA reconstruction performance using sparse measurement matrices as well as random measurement matrices. Finally a conclusion is provided in Section VII.

II. PRELIMINARIES

A. Compressed Sensing

The original compressed sensing problem is to recover a high-dimensional vector from a lower dimension set of linear equations. Consider a signal \( s \in \mathbb{R}^n \), and suppose that a certain basis \( \Psi \) provides a \( k \)-sparse representation \( x \) of \( s \) such that \( s = \Psi x \) with \( \|x\|_0 = k \). This vector \( s \) is not measured directly, but instead \( m \) projections of \( s \) are taken where \( m < n \). The projection vector is denoted by \( y \) where

\[
y = \Phi s
\]

with \( y \) an \( m \times 1 \) column vector and \( \Phi \in \mathbb{R}^{m \times n} \) being the measurement (projection) matrix. Since \( m < n \) the recovery of the signal \( s \) from \( y \) is, in general, not possible; however the assumption of a sparse signal \( x \) makes the reconstruction possible and practical.

As the sparse representation of \( s \) in the basis \( \Psi \) is \( x \), the recovery of \( s \) from \( y \) can be formulated as the \( \ell_0 \)-norm minimization problem on \( x \):

\[
\hat{x} = \arg\min_x \|x\|_0 \text{ s.t. } y = \Phi s = \Phi \Psi x.
\]

In other words we look for the sparsest vector \( x \) such that \( y = \Phi s = \Phi \Psi x \). This problem, sometimes called recovery via combinatorial optimization, is known to be NP-hard as the reconstruction algorithm needs to perform a combinatorial enumeration of all the \( \binom{n}{k} \) possible
sparse subspaces. Hence it cannot be used for practical applications.

Practically, a much easier problem yields an equivalent solution by solving for the minimum $\ell_1$-norm vector $x$ that agrees with the measurements $y$. This reconstruction is known as $\ell_1$-norm minimization, or recovery via convex optimization

$$\hat{x} = \text{argmin} \|x\|_1 \text{ s.t. } y = \Phi \Psi x.$$  \hfill (3)

This optimization is more tractable and can be solved by a linear programming technique called Basis Pursuit [11]. The complexity of this technique is cubic in $n$.

In the sequel, we simplify the notation, and consider only the reconstruction of the sparse vector $x$ from the observation vector $y$ with a measurement matrix $A$. The problem is then to solve

$$y = Ax.$$  \hfill (4)

In other words we either assume that the basis $\Psi$ is the identity matrix and $A = \Phi$, and hence our signal is sparse in the basis of the study, or that $\Phi \Psi = A$.

In [5] it has been shown that the $\ell_1$-norm minimization technique performs the exact reconstruction of the vector $x$ for a sufficient sparsity and a properly chosen measurement matrix $A$. The conditions on the measurement matrix were expressed in terms of the Restricted Isometry Property (RIP). Specifically, when the RIP is satisfied, a $k$-sparse vector $x$ can be reconstructed from a number of measurements $m$ on the order of $k \log (n/k)$. It is important to note that the RIP provides assurance that the output of the Basis Pursuit will be the same output as the $\ell_0$ minimization, however it can be different from the original vector $x$.

B. Iterative reconstruction methods

Soon after the seminal works of Candès, Tao, Romberg and Donoho, many studies were carried out to tackle the complexity of the $\ell_1$-minimization reconstruction technique. The main goal was to propose a variation of the LP solver providing good reconstruction performance with a lower complexity. Among all the proposed methods, we cite here some famous greedy algorithms like the Matching Pursuit [12], the Orthogonal Matching Pursuit (OMP)[13], [14], or Stagewise OMP [15]. The main idea of these greedy algorithms consists in the iterative approximation of the coefficients of the sparse signal to recover. They are known to be very fast and easy to implement, and guarantee the performance to be close to the original $\ell_1$-minimization. Many other algorithms exist in the literature (see Chapter 8 of [16] for an overview of the greedy algorithms used in compressed sensing).

The next section presents in detail one of the more well-known iterative reconstruction methods, namely approximate message-passing.

III. APPROXIMATE MESSAGE-PASSING ALGORITHM

Among the methods proposed to tackle the issue of the complexity of Basis Pursuit, message-passing algorithms have been introduced. Many of these algorithms have their roots in channel
coding applications. These algorithms primarily use the graphical representation of the measurement matrix to exchange information iteratively to recover the original vector \( x \). As for the Tanner graph [17] of low-density parity-check (LDPC) codes [18], the columns of the measurement matrix are associated with the variable nodes corresponding to the vector \( x \), and the rows of the measurement matrix correspond to the summation nodes, also called measurement nodes.

One of the first message-passing algorithms for compressed sensing was introduced by Donoho et al. in [8] for noise free measurements. The variables in message-passing algorithms are associated with edges in the bipartite graph representation of the matrix \( A \). Messages are updated according to the rules

\[
x^{(l+1)}_{v_i \rightarrow c_a} = \eta_l \left( \sum_{c_b \in N(v_i) \setminus c_a} a_{c_b,v_i} z^{(l)}_{c_b \rightarrow v_i} \right)
\]

\[
z^{(l)}_{c_a \rightarrow v_i} = y_{c_a} - \sum_{v_j \in N(c_a) \setminus v_i} a_{c_a,v_j} x^{(l)}_{v_j \rightarrow c_a}
\]

where \( c_a \) and \( c_b \) are measurement nodes, \( v_i \) and \( v_j \) are variable nodes, \( \eta_l \) is a sequence of threshold functions (applied componentwise), \( x^{(l)} \in \mathbb{R}^n \) is the current estimate of the solution \( x \), \( z^{(l)} \in \mathbb{R}^m \) is the current residual and \( N(v) \) is the set of neighbor nodes of the node \( v \) in the Tanner graph representation of \( A \). With a slight abuse in the notations, \( a_{c_a,v_j} \) represents the value at row \( a \) and column \( j \) of \( A \).

Donoho’s et al. [8] iterative thresholding algorithm, called the approximate message-passing algorithm (AMPA) follows from the message-passing algorithm given in (5) and (6). The authors of [8] argue that the right-hand side of Eq. (5) for update of messages \( x^{(l)}_{v_i \rightarrow c_a} \) does not depend strongly on the index \( a \) (specially if the matrix \( A \) is dense in which case only one out of \( n \) terms is excluded). They also argue that the right-hand side of Eq. (6) for updating \( z^{(l)}_{c_a \rightarrow v_i} \) does not depend strongly on \( i \). If these two dependencies are neglected, the messages are associated to graph vertices (\( x \) to variable nodes and \( z \) to measurement nodes). In contrast with the message-passing algorithm, which in general has to update \( nm \) messages, the update is done here on only \( n \) variable nodes and \( m \) measurement nodes.

Formally, the AMPA starts from an initial guess \( x^{(0)} = 0 \) and \( z^{(0)} = y \), and iteratively proceeds by calculating

\[
x^{(l+1)} = \eta_l(A^T z^{(l)} + x^{(l)})
\]

\[
z^{(l)} = y - A x^{(l)} + \frac{1}{\delta} z^{(l-1)} \langle \eta'_{l-1}(A^T z^{(l-1)} + x^{(l-1)}) \rangle,
\]

where \( \eta_l \) is a sequence of threshold functions (applied componentwise), \( x^{(l)} \in \mathbb{R}^n \) is the current estimate of the solution \( x \), \( A^T \) denotes the transpose of \( A \) and \( \eta'(u) = \partial \eta(u)/\partial u \). The bracket \( \langle \rangle \) operator applied on a vector \( v = (v_i)_{1 \leq i \leq n} \) gives \( \langle v \rangle = (1/n) \sum_{1 \leq i \leq n} v_i \). The role of the additional term in the update of \( z^{(l)} \) is to cancel the correlation between the current vector estimates and their past values. A typical thresholding function \( \eta \) is the soft thresholding given by

\[
\eta(x; \lambda) = \text{sgn}(x)(|x| - \lambda)_+
\]

where the subscript \( (u)_+ = \text{sgn}(u)(|u|) \).
As mentioned in [8], the AMPA is suitable when the entries $a_{i,j}$ of the matrix $A$ are independent and identically distributed. The measurements need then to be acquired randomly to be able to use the AMPA. It has been shown that the reconstruction possibilities can be established through the phase-transition diagram which basically provide the bounds on $k$, $n$, $m$ so that the AMPA is able to recover $k$-sparse signals.

The next section provides a full presentation of the interval-passing algorithm (IPA) used together with the AMPA in our simulation results.

IV. INTERVAL-PASSING ALGORITHM

Chandar et al. [9] introduced a simple message-passing algorithm for reconstructing non-negative signals using sparse binary measurement matrices. We modified this algorithm in order to deal with non-negative real-valued measurement matrices and refer to it as IPA [10]. From [9], the complexity of the algorithm is $O(n(\log(\frac{n}{k}))^2 \log(k))$ which is a good trade-off between the polynomial complexity of the LP reconstruction, and the linear complexity of the simple verification decoding [19], which we have shown in [10] through reconstruction performance. The use of sparse measurement matrices originates from the work of Dimakis and Vontobel [20], who studied the use of LDPC-based measurement matrices in compressed sensing and showed that good LDPC matrices used for LP decoding are also good for LP reconstruction. Besides, the use of sparse measurement matrices enables a low-complexity measurement process.

Like the AMPA, the IPA is a message-passing algorithm, and thus messages are associated with the graphical representation of the measurement matrix to perform the reconstruction. Let $V = \{v_1, v_2, ..., v_n\}$ and $C = \{c_1, c_2, ..., c_m\}$ be respectively the sets of variable nodes and measurement nodes in the graphical representation of the measurement matrix $A = \{a_{j,i}\}$ for $1 \leq j \leq m$ and $1 \leq i \leq n$. The graphical representation of $A$ is actually the Tanner graph [17] of the binary image\(^1\) of $A$, whose edges are labeled by the real values corresponding to the non-zero positions in $A$.

In the IPA, the messages passing through edges are intervals $[\mu, M]$ corresponding to the lower and upper bounds of the estimation of the connected variable node. At each iteration $l$, the message update from the variable $v_i$ to the measurement node $c_j$ is given by:

\[
\mu_{v_i \rightarrow c_j}^{(l)} = \max_{c_j' \in \mathcal{N}(v_i)} \left( \mu_{c_j' \rightarrow v_i}^{(l-1)} \right) \times a_{j,i} \tag{10}
\]

\[
M_{v_i \rightarrow c_j}^{(l)} = \min_{c_j' \in \mathcal{N}(v_i)} \left( M_{c_j' \rightarrow v_i}^{(l-1)} \right) \times a_{j,i} \tag{11}
\]

and the messages from the measurement node $c_j$ to the variable node $v_i$ are updated as:

\(\text{A matrix } H = \{h_{j,i}\} \text{ is said to be the binary image of a matrix } A = \{a_{j,i}\} \text{ if } h_{j,i} = 1 \text{ if } a_{j,i} \neq 0 \text{ and } h_{j,i} = 0 \text{ if } a_{j,i} = 0.\)
\[
\mu_{c_j \rightarrow v_i}^{(l)} = \max \left\{ 0, \frac{y_j - \sum_{v'_i \in \mathcal{N}(c_j) \setminus \{v_i\}} M_{v'_i \rightarrow c_j}^{(l)}}{a_{j,i}} \right\}
\]

\[
M_{c_j \rightarrow v_i}^{(l)} = \frac{y_j - \sum_{v'_i \in \mathcal{N}(c_j) \setminus \{v_i\}} \mu_{v'_i \rightarrow c_j}^{(l)}}{a_{j,i}}
\]

where \(\mathcal{N}(v_i)\) (resp. \(\mathcal{N}(c_j)\)) is the set of measurement (resp. variable) nodes which are the neighbors of \(v_i\) (resp. \(c_j\)) in the Tanner graph of \(A\). The message update is shown in Fig. 1.

The reconstruction process stops when the maximum number of iterations is reached, or the lower bound and the upper bound of the interval from variable nodes to measurement nodes has converged to a common value for every variable node. This common value is set as the estimate of each connected variable node value. When the lower bound and the upper bound do not converge, we arbitrarily set the estimate value to the lower bound.

V. APPLICATIONS OF COMPRESSED SENSING

The applications of compressed sensing are numerous, as the assumption of sparsity can be relevant in many areas of signal processing. For instance, the wavelet decomposition [21] generally provides a sparse approximation of natural images, such as the example of Fig. 2. The majority of the coefficients of the wavelet representation have a small absolute value and can be considered as zero or near zero. Thus in the wavelet domain, the data can be considered as sparse, or pseudo-sparse. Using sparse or random measurement matrices on the wavelet transform of an image can lead to compression using AMPA or IPA.
Other domains of signal processing are concerned with compressed sensing, such as radar [22], imaging [23], digital communications [24] or even biomedical [25], so long as a sparse representation of a signal exists in some basis. Applications of compressed sensing can also be found in genetics [16], in which \( n \) genes are studied while analyzing \( m \) patients. Commonly only a few genes will be active for each patient, hence the sparsity.

More recently, we have shown [26] that compressed sensing can be used in the chemical mixture estimation problem in spectroscopy. We first assume that we know perfectly the spectrum of a set of pharmaceutical chemicals that can be present in a surface or in a solution. In a chemical mixture we assume that only few chemicals can be present among this set, hence the sparseness of the vector representing the chemical mixture. By using an appropriate measurement matrix on the spectrum of a chemical mixture, we can achieve impressive compression ratios (8:1 for example). By using the IPA presented above we ensure a low-complexity reconstruction process while maintaining very high reconstruction performance.

An important database of papers on compressed sensing and its application can be found in [27]. Up-to-date articles and papers, as well as some interesting discussions, can be found in [28].

VI. SIMULATION RESULTS

In this section we present the reconstruction performance of the two algorithms presented above. We produce a \( k \)-sparse vector of length \( n \) as follows. For a given sparsity \( k \), we generate a non-negative vector of length \( k \) (entries of this vector are drawn from the uniform distribution on 0 to 1) and then assign randomly each of the \( k \) generated values among the \( n \) values of \( x \). The measurement vector \( y \) is then obtained via \( y = Ax \). For a given \( n \) and \( m \) the measurement matrix \( A \) is either a random matrix whose elements are independent and identically distributed drawn from a Gaussian distribution, or deterministic (more precisely, sparse and based on LDPC code design). For the cases reported below, we perform 75 reconstructions for each sparsity \( k \).
and display the proportion of correct reconstruction as a function of $k$. A vector is said to be correctly recovered if each $x_j$, $j = 1, \ldots, n$ is recovered with absolute value no greater than $10^{-6}$ for the IPA reconstruction and no greater than $10^{-3}$ for the AMPA reconstruction. The AMPA does not allow, in general, for perfect recovery (in the sense of the IPA) which explains why we relax the criterion to $10^{-3}$.

We first report the reconstruction results in the noise free case using $n = 1008$ and a fixed $m = 504$ in Fig. 3. This gives a compression ratio of two. The measurement matrix for the IPA was chosen to be the one from the Gallager codes of rate half and column-weight-three in [29]. One can see that the exact recovery can be achieved up to a sparsity of 175 for the IPA, whereas the AMPA is only accurate enough for a sparsity of 80, even with the relaxed criterion for correctness.

As a second example we report the comparison of the AMPA and the IPA for a fixed signal length $n = 265$ and two different values for $m$. The random matrices for the AMPA were designed as previously. The deterministic matrices for IPA are described in what follows. These measurement matrices $M$ originate from the code construction technique of structured LDPC matrices [30] where each matrix $M$ is a block matrix of circulants. We use binary measurement matrices for the sake of simplicity. The use of non-negative measurement matrices give similar reconstruction results [10]. For the first example, let $p = 53$ and let:

$$M_1 = \begin{pmatrix} I_0 & I_0 & I_0 & I_0 & I_0 \\ I_0 & I_{14} & I_1 & I_{36} & I_{37} \\ I_0 & I_3 & I_4 & I_{38} & I_{42} \end{pmatrix}$$

(14)

where $I_j$ denotes the $p \times p$ identity matrix circularly shifted by a factor $j$. This measurement matrix $M_1$ is of size $m \times n = 3p \times 5p = 159 \times 265$. Similarly, the second matrix $M_2$ we use in this paper is of size $m \times n = 4p \times 5p = 212 \times 265$ and is given by:

$$M_2 = \begin{pmatrix} I_0 & I_0 & I_0 & I_0 & I_0 \\ I_0 & I_{34} & I_{39} & I_{49} & I_{48} \\ I_0 & I_{25} & I_9 & I_{27} & I_{51} \\ I_0 & I_{28} & I_4 & I_7 & I_{15} \end{pmatrix}.$$  

(15)
We can see in Fig. 4 that the reconstruction performance for the IPA is always superior to the performance for the AMPA. For instance, the AMPA exhibits failures whenever $k \geq 30$ for both choices of the number of measurement taken. On the other hand, the IPA does not exhibit failures until $k \geq 55$ and $m = 159$, and $k \geq 75$ for $m = 212$. The improvement in performance appears to be slightly less that the example of Fig. 3. The explanation for this difference, as well as the failures of the IPA/AMPA is out of the scope of this paper, but we can infer that the length of the signal $x$ has a strong influence. The failures of the IPA (studied in [10]) as well as the phase transition diagram of the AMPA could also explain this difference. This discussion and related problems will be addressed in the journal version of this paper.

VII. CONCLUSION

In this paper we have discussed the emerging field of signal processing known as compressed sensing. We presented two low-complexity iterative reconstruction algorithms, namely IPA and AMPA, which have appealing reconstruction performance in a noise free environment. The IPA has more accurate reconstruction capability, while the reconstruction possibilities of the AMPA can be studied theoretically.

Future work includes the description of the noisy case. The performance metric will be the mean-square error. Preliminary results show that with a modification of the IPA, good mean-square error reconstruction performance can be obtained. In the noisy case the decision step of the IPA has to be modified, as the lower and upper bounds will not converge. The AMPA does not require modification and may be more suitable.

VIII. ACKNOWLEDGEMENTS

The authors thank V. Ravanmehr for her work on the IPA. This work was funded by DARPA under the Knowledge Enhanced Compressive Measurements (KECoM) program through contract
REFERENCES


#N66001-10-1-4079 and by the NSF under grant CCF-0963726.