Erasure-correcting vs. erasure-detecting codes for the full-duplex binary erasure relay channel

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***Abstract*—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. In this paper, the asymptotic performance of backward and joint iterative decoders over the block-Markov (BM) structure based on sparse-graph codes is investigated. We show that there the BM structures based on good error-correcting codes or good error-detecting codes have a different behavior, although both of them can be used to come close to the theoretic limit of the binary erasure relay channel (BE-RC).**

I. Introduction

Over the last decade, coding for cooperative communications using the Decode-and-Forward strategy has received a growing attention from both the coding theory and communication theory communities. Since the early works on distributed turbo-codes [1], they have been numerous coding schemes that have been proposed in different contexts, ie. considering different duplexing mode (half- (HD) vs full-duplex (FD) communications) or addressing communications over different channels (e.g. erasure, Gaussian or Rayleigh-fading channels) [1]-[12]. Among the proposed coding schemes, sparse-graph codes based coding schemes have been mainly considered, and their performances have been mainly evaluated over HD relay channels (in both othorgonal and non-orthogonal multiple accesses to the destination). For sparse-graph based codes, an efficient coding approach has been proposed in [7], that is refered to as bilayer Low-Density Parity-Check (LDPC) codes. This coding approach has given rise to numerous extensions or improvements (see [10] and references therein). Another recent work is [12], where spatially-coupled LDPC codes for orthogonal erasure relay channels were considered.

Recently, we performed the asymptotic analysis of bilayer sparse-graph-based codes for the block-Markov (BM) DF scheme, in both HD and FD regimes [13]. Joint iterative decoding and backward decoding algorithms were considered. We observed in [13] that the asymptotic performance of the block-Markov structure, based on bilayer LDPC codes, was worse than the asymptotic performance of the non-block-Markov scheme. Given that a block-Markov structure is implicitly presumed in the non-orthogonal scenario, does it mean that sparse-graph codes will perform badly in the non-orthogonal regime? The aim of the present paper is to respond to this question and to study in which ways one could attain the theoretical limit in the non-orthogonal scenario. The orthogonal scenario has also been considered.

II. Model

In this paper, we consider a full-duplex relay channel that consists of the source S, communicating with the destination D, with the help of the node R. At R, the Decode-and-Forward transmission protocol is assumed[14] (R receives the source message, decodes, re-encodes and re-sends it to D. We distinguish two reception scenarios: orthogonal (OR), when D receives two messages from S and R separately, and non-orthogonal (NOR), when D receives a mix of messages.

The generation of two messages (X_1 at S and X_2 at R) is based on the block-Markov encoding scheme as given in [13]. Let w_i be the i-th information message $i = 1 \dots M$. At S, w_i is encoded using a code C_S into the message $X_1(w_i)$. The message sent by the source is formed as follows

$$X_1(w_i|s_i) = X_1(w_i) \oplus X_2(s_i), \qquad (1)$$

where s_i is the bin number corresponding to the information mesage w_{i-1} and $X_2(s_i)$ is a codeword of a code C_R.

This encoding scheme is associated with the following transmission protocol:

- Let $s_1 = 0$.
- For i from 1 to M, repeat:
  - $X_1(w_i)$ is generated from C_S at the source end and $X_1(w_i|s_i)$ is broadcasted.
  - R receives $Y_1(i)$, a noisy version of $X_1(w_i|s_i)$, estimates $X_1(w_i)$ and computes s_{i+1}. Then $X_2(s_{i+1})$ is generated from C_R and is sent during the next transmission time;
  - at time i, in the non-orthogonal scenario, D receives $Y(i)$, based on $X_1(w_i|s_i)$ and on $X_2(s_i)$, while in the orthogonal case, D receives Y_S and Y_R, noisy versions of $X_1(w_i|s_i)$ and on $X_2(s_i)$, separately.
- For $i = M + 1$, R sends $X_2(s_M)$ while S broadcasts w_{M+1} set to 0.

For the sake of simplicity, we assume the transmission over the binary erasure relay channel (BE-RC), with erasure probability ϵ_{SD} on the link SD and erasure probability ϵ_{RD} on the link RD. In the NOR regime, the channel at D

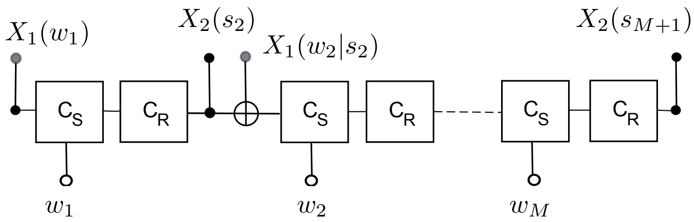

Fig. 1. Block-Markov structure to encode M information messages w_i.

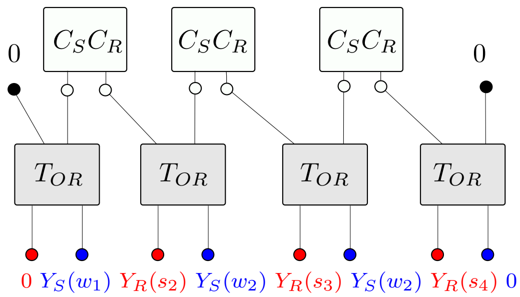

Fig. 2. BM structure in orthogonal (OR) scenario with $M = 3$.

is modelled as the sum modulo-2 multiple access channel, already considered in [13].

The transmission protocol above provides the block-Markov (BM) structure depicted in Fig.1. C_S and C_R blocks represent coding schemes at S and at R. This structure can be decoded using numerous decoding schedules. In this paper we consider *joint iterative decoder* that consists in the following:

- at update $\mathtt{l}$, $1 \leq \mathtt{l} \leq L$, one forward and one backward steps are performed, during which estimates of $\hat{X}_1^{(\mathtt{l})}(w_i|s_i)$ and $\hat{X}_2^{(\mathtt{l})}(s_{i+1})$ are calculated, for all $1 \leq i \leq M$, given backward and forward estimations at the previous update[1];
- the final estimate $\hat{w}_i$ is computed for all $1 \leq i \leq M$, given $\hat{X}_1^{(L)}(w_i|s_i)$ and $\hat{X}_2^{(L)}(s_{i+1})$.

A *backward decoder* will be also considered. The backward decoder can be seen as the first backward step of the joint iterative decoder.

Let us define the decoding convergence region:

Definition 1: The decoding (per code block, backward or joint iterative) is assumed to be *successful* if all corresponding information messages w_i $\forall i \in [1, M]$ have been successfully recovered. Hence the convergence region of the corresponding decoder is the set of all input parameters such that the decoding is successful.

In what follows, the asymptotic decoding performance of backward and joint iterative decoders over BM structures is investigated.

III. Forward/Backward Decoding Updates over the BM Structure

The BM structure can be seen as a sequence of alternating blocks $C_S - C_R$ ("coding blocks"), representing a joint coding

[1] i an M denote the index and the total number of information messages w_i, and $\mathtt{l}$ and L denotethe current and the maximum number of forward-backward updates.

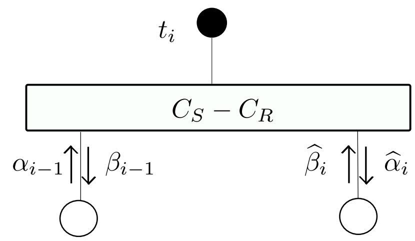

Fig. 3. The i-th block $C_S - C_R$, related input/output forward/backward and information erasure probabilities.

scheme at S and at R, and T ("channel block"), representing the channel function. An example of this representation for the OR scenario is given in Fig.2. In this section we are going to describe forward/backward updates over the BM structure, that make part of backward and joint iterative algorithms.

A. $C_S - C_R$ Block and Update Functions $F^{\mathtt{fwd}}/F^{\mathtt{bwd}}$

First, let us consider a block $C_S - C_R$.

Definition 2: For a code block i, let $\alpha_{i-1}/\widehat{\alpha}_i$ and $\widehat{\beta}_i/\beta_{i-1}$ be average forward and backward input/output erasure probabilities, corresponding to $X_1(w_i)/X_2(s_i)$, as it is shown in Fig.3. Hence we define the forward/backward update function:

Definition 3: At the forward/backward update $\mathtt{l}$ for block i, the forward/backward code functions $F_{\widehat{\beta}}^{\mathtt{fwd}}/F_{\alpha}^{\mathtt{bwd}}$, parametrized by $\widehat{\beta}_i^{(\mathtt{l}-1)}/\alpha_i^{(\mathtt{l}-1)}$,are defined as follows:

$$F_{\widehat{\beta}}^{\mathtt{fwd}}: \quad \widehat{\alpha}_i^{(\mathtt{l})} = F_{\widehat{\beta}}^{\mathtt{fwd}}(\alpha_{i-1}^{(\mathtt{l})}); \tag{2}$$

$$F_{\alpha}^{\mathtt{bwd}}: \quad \beta_{i-1}^{(\mathtt{l})} = F_{\alpha}^{\mathtt{bwd}}(\widehat{\beta}_i^{(\mathtt{l})}); \tag{3}$$

Definition 4: Let t_i denote the decoding erasure probability at information positions of the code block i. Then we define the information update function $F^{\mathtt{inf}}(\alpha, \widehat{\beta})$ so that $t_i = F^{\mathtt{inf}}(\alpha_{i-1}^{(\mathtt{l})}, \widehat{\beta}_i^{(\mathtt{l})})$.

Proposition 1: It can be shown that the following holds:
1) $F_x(y)$ is non-decreasing both with x with y;
2) $F_0(0) = 0$ and $F_1(1) = 1$;
3) $F_b^{\mathtt{fwd}}(a) \geq F^{\mathtt{inf}}(a, b)$ and $F_b^{\mathtt{bwd}}(a) \geq F^{\mathtt{inf}}(a, b)$.

Notation 1: Let us denote the convergence region of a $C_S - C_R$ block by Γ, and the boundary of this region by γ. Note that one can define boundaries related to one forward/backward update as follows:

$$\gamma^{fwd}(\beta) = \max_{0 \geq \alpha \geq 1} \{\alpha : F_{\beta}^{\mathtt{fwd}}(\alpha) = 0\},$$

$$\gamma^{bwd}(\alpha) = \max_{0 \geq \beta \geq 1} \{\beta : F_{\alpha}^{\mathtt{fwd}}(\beta) = 0\}.$$

B. T block and Update Functions $G^{\mathtt{fwd}}/G^{\mathtt{bwd}}$

Let erasure probabilities of S-D and R-D links be ϵ_{SD} and ϵ_{RD} respectively.

Definition 5: At update $\mathtt{l}$, we define forward/backward channel functions as follows (see Fig.4 for illustration):

$$G_{\epsilon_{SD},\epsilon_{RD}}^{\mathtt{fwd}}: \quad \alpha_i^{(\mathtt{l})} = G_{\epsilon_{SD},\epsilon_{RD}}^{\mathtt{fwd}}(\widehat{\alpha}_i^{(\mathtt{l})}), \tag{4}$$

$$G_{\epsilon_{SD},\epsilon_{RD}}^{\mathtt{bwd}}: \quad \widehat{\beta}_i^{(\mathtt{l})} = G_{\epsilon_{SD},\epsilon_{RD}}^{\mathtt{bwd}}(\beta_i^{(\mathtt{l})}). \tag{5}$$

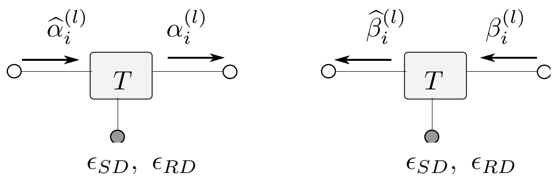

Fig. 4. The input/output forward/backward erasure probabilities of i-th block T_{OR}/T_{NOR}.

In what follows, the subscript $(\epsilon_{SD}, \epsilon_{RD})$ is omitted for simplicity.

Note that, Ifor the BE-RC case, $G^{\mathtt{fwd}}(y)$ and $G^{\mathtt{bwd}}(y)$ are:

1) Orthogonal reception (OR) scenario:

$$G^{\mathtt{fwd}}(y) = 1 - (1 - \epsilon_{SD})(1 - \epsilon_{RD} y) \quad (6)$$

$$G^{\mathtt{bwd}}(y) = \epsilon_{RD}(1 - (1 - \epsilon_{SD})(1 - y)) \quad (7)$$

2) Non-orthogonal reception (NOR) scenario:

$$G^{\mathtt{fwd}}(y) = \epsilon_{SD}\bar{\epsilon}_{RD} + \epsilon_{SD}\epsilon_{RD} + \bar{\epsilon}_{RD}\epsilon_{SD} y \quad (8)$$

$$G^{\mathtt{bwd}}(y) = \bar{\epsilon}_{SD}\bar{\epsilon}_{RD} + \epsilon_{SD}\epsilon_{RD} + \bar{\epsilon}_{SD}\epsilon_{RD} y \quad (9)$$

C. Expressions of Forward/Backward Updates for the Joint Iterative Decoder

Let us consider the joint iterative decoder[2]. Note that the density evolution for BM structures, together with forward/backward updates, has been derived in [13]. Let us present it by means of functions defined above. Recall that the first message of R and the last message of S are known at D, so initial forward/backward erasure probabilities are:

$$\alpha_i^{(0)} = \begin{cases} G^{\mathtt{fwd}}(1), & \text{if } 2 \leq i \leq M, \\ G^{\mathtt{fwd}}(0), & \text{if } i = 1, \end{cases} \quad (10)$$

$$\widehat{\beta}_i^{(0)} = \begin{cases} G^{\mathtt{bwd}}(1), & \text{if } 1 \leq i \leq M-1, \\ G^{\mathtt{bwd}}(0), & \text{if } i = M, \end{cases} \quad (11)$$

and the forward/backward update $\mathtt{1} > 0$ can be written as:

$$\alpha_i^{(1)} = G^{\mathtt{fwd}}\left(F^{\mathtt{fwd}}_{\widehat{\beta}_i^{(1-1)}}(\alpha_{i-1}^{(1)})\right); \quad (12)$$

$$\widehat{\beta}_i^{(1)} = G^{\mathtt{bwd}}\left(F^{\mathtt{bwd}}_{\alpha_{i+1}^{(1-1)}}(\widehat{\beta}_{i+1}^{(1)})\right). \quad (13)$$

Remark 1: The density evolution presented above can be represented graphically (Fig.5), similar to EXIT charts [15].

Notation 2: Denote by Γ^{BWD} and Γ^{JNT} the convergence regions of backward and joint iterative decoders.

D. Two Classes of Codes: Erasure-Correcting (EC) Codes and Erasure-Detecting (ED) Codes

Let us consider some interesting classes of forward functions[3] $F^{\mathtt{fwd}}$. Consider the asymptotic iterative performance of a $C_S - C_R$ block. Then we consider two cases:

(a) $F^{\mathtt{fwd}} = \begin{cases} \approx 1, & \epsilon^* \leq \epsilon^* \leq 1 \\ 0, & \epsilon < \epsilon^* \end{cases}$;

[2]The backward decoder is a particular case of the joint iterative one.

[3]Or, similarly, of $F^{\mathtt{bwd}}$.

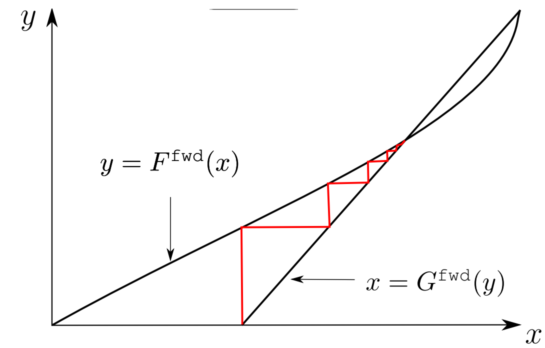

Fig. 5. A graphical representation of the forward evolution.

(b) $F^{\mathtt{fwd}}$ is strictly increasing with ϵ, $\epsilon \in [0, 1]$.

Note that F functions of basically all efficient erasure-correcting codes (e.g. LDPC, turbo codes) belong to the class (a). The class (b) represents the class of erasure-detecting codes, which are typically bad erasure-correctors (e.g. LDGM). Hence let us denote these classes as Erasure-Correcting class of codes (EC) and Erasure-Detecting class of codes (ED), and evaluate their performance in the block-Markov encoding structure.

IV. Block-Markov Structure Based on EC Codes

We have the following important lemma:

Lemma 1: For a BM structure based on EC codes,

$$\Gamma^{\mathrm{BWD}} = \Gamma(G^{\mathtt{fwd}}(1), G^{\mathtt{bwd}}(0)),$$

$$\Gamma^{\mathrm{JNT}} = \Gamma(G^{\mathtt{fwd}}(1), G^{\mathtt{bwd}}(0)). \cup \Gamma(G^{\mathtt{fwd}}(0), G^{\mathtt{bwd}}(1))$$

Proof: Consider the forward update (the backward case is similar). Note that, for EC codes:

$$F^{\mathtt{fwd}}_{\widehat{\beta}}(\alpha) \approx \mathbb{1}(\alpha > \gamma^{fwd}(\widehat{\beta})),$$

$$F^{\mathtt{bwd}}_{\alpha}(\widehat{\beta}) \approx \mathbb{1}(\widehat{\beta} > \gamma^{bwd}(\alpha)).$$

Consider the case when $F^{\mathtt{fwd}}_{\widehat{\beta}_0}(\alpha_0^{(1)}) = F^{\mathtt{fwd}}_{G^{\mathtt{bwd}}(1)}(G^{\mathtt{fwd}}(0)) = 0$ (bits of $X_2(s_1)$ are decoded perfectly). Then $\alpha_1^{(1)} = G^{\mathtt{fwd}}(0)$, implying $F^{\mathtt{fwd}}_{\widehat{\beta}_0}(\alpha_i^{(1)}) = 0$, for $i = 1 \ldots M$. As $F^{\mathtt{inf}}(\alpha_1, \widehat{\beta}_0) \leq F^{\mathtt{fwd}}_{\widehat{\beta}_0}(\alpha_i^{(1)}) = 0$, the decoder converges in one forward update.

Now let $G^{\mathtt{fwd}}(0) > \gamma^{fwd}(G^{\mathtt{bwd}}(1))$ (bits of $X_2(s_1)$ remained completely unknown and $\alpha_1^{(1)} = G^{\mathtt{fwd}}(1)$). By non-decreasing property of both $G^{\mathtt{fwd}}$ and $F^{\mathtt{fwd}}$ we get that $F^{\mathtt{fwd}}_{\widehat{\beta}_0}(\alpha_i^{(1)}) = 1$, for $i = 1 \ldots M$ and decoding fails. ■

The theorem below follows directly Lemma 1.

Theorem 1: Convergence regions of the BM structure based on EC codes is bounded by the convergence region of one $C_S - C_R$ block, i.e.

$$\Gamma^{\mathrm{BWD}} \subseteq \Gamma \quad \text{and} \quad \Gamma^{\mathrm{JNT}} \subseteq \Gamma.$$

Thus in order to design a capacity-approaching BM structure based on EC codes, one should necessarily choose a capacity-approaching $C_S - C_R$ code.

V. Block-Markov Structure Based on ED Codes

Similarly to Lemma 1, for ED codes:

Lemma 2: Assuming that the BM decoding process converges sufficiently fast, threshold boundaries of the BM structure based on an ED ensemble are given by:

$$\Gamma^{\text{BWD}} = \Gamma(G^{\texttt{fwd}}(1), G^{\texttt{bwd}}(0)), \quad (14)$$
$$\Gamma^{\text{JNT}} = \Gamma(\alpha_\infty^{(\infty)}, \beta_\infty^{(\infty)}). \quad (15)$$

The proof follows directly from definition of $F^{\texttt{fwd}}$ and F^{bwd} for ED codes.

From Lemma 2 the following result follows directly:

Theorem 2: Convergence regions of the BM structure based on ED codes have the following property:

$$\Gamma^{\text{BWD}} \supseteq \Gamma \quad \text{and} \quad \Gamma^{\text{JNT}} \supseteq \Gamma.$$

We obtain an interesting result here: the iterative performance of the BM structure of ED codes may outperform the performance of one erasure-detecting $C_S - C_R$ block! The question is whether it can attain the theoretical limit for any value of ϵ_{SD}? Unfortunately, the answer is no. To show it, let us consider two extreme points of the threshold boundary for the BM structure based on ED codes: $\epsilon_{SD} = 1$ and $\epsilon_{RD} = 1$.

Theorem 3: The block-Markov structure based on ED codes is bounded away from capacity for $\epsilon_{SD} = 1$.

Proof: Without loss of generality, let us consider the orthogonal (OR) scenario. For $\epsilon_{SD} = 1$, channel functions $G^{\texttt{fwd}}$ and G^{bwd} can be written as $G^{\texttt{fwd}}(y) = 1$ and $G^{\text{bwd}}(y) = \epsilon_{RD}$, implying that the BM structure is broken into separate blocks $C_S - C_R$. Therefore, the iterative threshold of the BM structure is equal to the one of the block $\bar{C}_S - C_R$. ∎

Note that a different thing happens in the region ϵ_{RD} close to 1: in this case, channel functions are given by $G^{\texttt{fwd}}(y) = G^{\text{bwd}}(y) = 1 - \bar{\epsilon}_{SD}\bar{y}$, and the BM structure may be capacity-approaching.

A. Example of a BM Ensemble Based on ED Codes

To illustrate our analysis, let us give an example of a BM code, based on LDGM codes:

Definition 6: Bilayer Lengthened LDGM (BL-LDGM) codes are bilayer $C_S - C_R$ codes such that C_S and C_R are two LDGM codes, sharing their information bits and having respective degree distributions $(\lambda_S(x), \rho_S(x))$ and $(\lambda_R(x), \rho_R(x))$.

Notation 3: Let us denote by BM-BL-LDGM the BM structure based on bilayer LDGM codes.

Let us optimize distributions $\lambda_S(x)$ and $\lambda_R(x)$ (for simplicity, a constant check node degree is assumed). We define the following optimization problem that maximizes the rate of the BL-LDGM code:

$$max \quad \sum_{i=1}^{i_{max}} \lambda_{S,i}/i \quad (16)$$
$$\text{with} \quad (\alpha_\infty^{(\infty)}, \beta_\infty^{(\infty)}) \in \Gamma$$
$$\sum_{i=1}^{i_{max}} \lambda_{S,i} = 1 \ \text{ and } \ \lambda_{S,i}, \geq 0, \quad \forall\, 1 \leq l \leq i_{max}.$$

This is a non-linear optimization problem, and it can be solved numerically.

Example 1: We start by fixing C_R to be a regular (x^4, x^4) LDGM ensemble. It has an iterative decoding threshold $\epsilon^* = 0.313$. We are going to optimize $\lambda_R(x)$ by the optimization algorithm stated above, using the blind random search optimization [16]. The obtained degree distribution is

$$\begin{aligned} \lambda_S(x) &= 0.0481x + 0.048x^2 + 0.0199x^3 + 0.0196x^4 \\ &+ 0.046x^5 + 0.0441x^6 + 0.1029x^7 + 0.0639x^8 \\ &+ 0.061x^9 + 0.0172x^{10} + 0.1163x^{11} + 0.0885x^{12} \\ &+ 0.1096x^{13} + 0.0861x^{14} + 0.1291x^{15} \end{aligned}$$

The threshold boundary of the BM structure with the optimized C_R part is shown in Fig.6: dashed lines represent the orthogonal reception (OR), and full lines represent the non-orthogonal (NOR) one. Black curves correspond to threshold boundaries; achievable rates region are given by grey lines. Note that for $\epsilon_{RD} = 1$, theBM-BL-LDGM ensemble has multiplicative gap to capacity $\delta \approx 0.003$, due to the optimization of the degree distribution. For $\epsilon_{SD} = 1$, the gap to capacity is around 0.35 (the extreme point corresponds to the iterative threshold of the LDGM serving as C_R). Note that obtained results are consistent with our analysis.

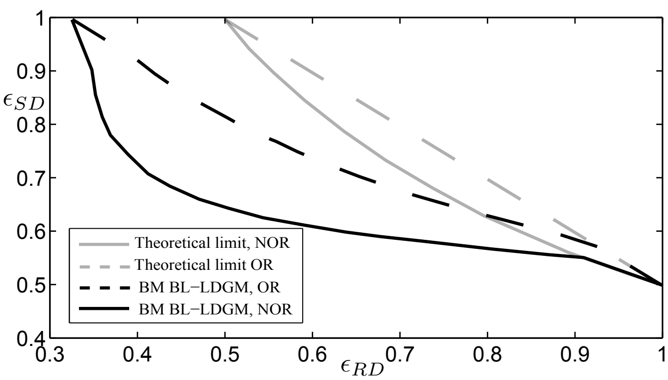

Fig. 6. Threshold boundaries of the $(\lambda(x), \rho(x))$ BM-BL-LDGM ensemble from Example 1, in OR and NOR regimes.

VI. Our New Construction

We would like to design a new $C_S - C_R$ code family which would be capacity-approaching at two points $\epsilon_{SD} = 1$ and $\epsilon_{RD} = 1$ simultaneously, both in orthogonal and non-orthogonal scenarios. We propose the following code structure:

Definition 7: Bilayer-Expurgated LDGM (BE-LDGM) codes are bilayer $C_S - C_R$ codes such that C_S and C_R codes are given respectively by an LDGM and an LDPC codes, that share their information bits. Corresponding degree distributions are denoted by $(\lambda_S(x), \rho_S(x))$ and $(\lambda_R(x), \rho_R(x))$.

We develop the following design condition:

Proposition 2: Consider a BE-LDGM ensemble, and let ϵ^*_{LDPC} be the iterative threshold of the constituent LDPC part. The BE-LDGM ensemble simultaneously approaches the

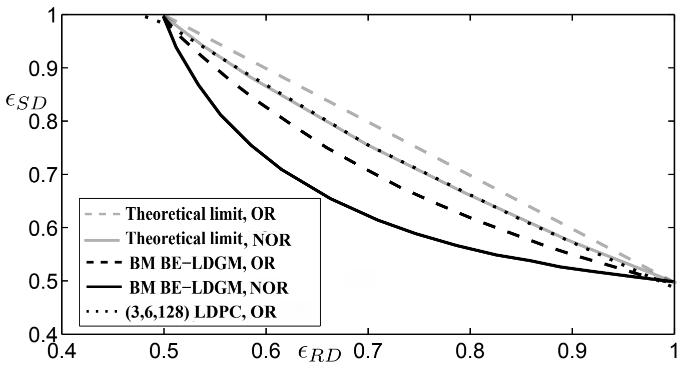

Fig. 7. Threshold boundaries of the BM structure based on BE-LDGM code.

theoretical limit in both points $\epsilon_{SD} = 1$ and $\epsilon_{RD} = 1$ if the following is verified:

$$\epsilon^*_{LDPC} = \epsilon^{th}_{SD}, \quad (17)$$
$$\epsilon^*_{LDPC} \geq \epsilon^{th}_{RD} L_S(1 - (1 - \epsilon^{th}_{SD})\rho_S(1 - \epsilon^{th}_{SD})), \quad (18)$$

where ϵ^{th}_{RD} and ϵ^{th}_{SD} denote Shannon limits for R-D and S-D links, and $L_S(x)$ is the node-perspective degree distribution of variable nodes for the C_S ensemble (i.e. LDGM).

Proof: For $\epsilon_{SD} = 1$, the performance of the BE-LDGM ensemble is conditioned by the performance of its LDPC part (the LDGM is not able to decode at all), so (17) follows directly. Consider now $\epsilon_{RD} = 1$ and notice that, if the decoding of the LDGM part has the output probability below ϵ^*_{LDPC}, the overall decoding of the BE-LDGM code succeeds. ■

Let us give an example of a BE-LDGM ensemble, satisfying (17)-(18).

Example 2: Choose the C_R part to be an LDPC ensemble with a constant check degree and $\lambda(x)$ taken from [17]. Such ensemble has a small gap to capacity ($\delta = 10^{-3}$) over the BEC. Choose the C_S part to be the (x^4, x^4) LDGM ensemble. It can be checked that both (17) and (18) are verified.

Fix $M = 100$ and consider a full-duplex BE-RC, in both orthogonal (OR) and non-orthogonal (NOR) regimes. The threshold boundaries for our ensemble are shown in Fig.7: dashed lines represent OR, and full lines represent NOR. For comparison, the dotted curve represents the boundary of spatially-coupled (3,6,128) LDPC ensemble from [12], which is one of efficient codes over BE-RC in the OR regime.

At $\epsilon_{RD} = 1$, the gap to capacity is $\delta \approx 10^{-5}$, and $\delta = 10^{-3}$ at $\epsilon_{SD} = 1$.

VII. Acknowledgement

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VIII. Conclusion

In this paper, we have considered asymptotic convergence regions of the block-Markov structure for the binary erasure relay channel, based on EC or ED codes, both for orthogonal and non-orthogonal reception at the destination. It has been shown that for EC codes, the BM performance is limited the performance of one block in the BM structure For ED codes, the BM performance may outperform the performance of one block. It has been also shown that the performance of a BM-ED code cannot be improved over ϵ_{RD} and ϵ_{SD} simultaneously. A new construction for the $C_S - C_R$ block to overcome this limitation has been proposed. The new BM schemes based on the new family perform close to the theoretical limit. The question whether the new codes can be capacity-approaching is still open.

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