Erasure-correcting vs. erasure-detecting codes for the full-duplex binary erasure relay channel

Marina Ivashkina  
ETIS group  
ENSEA/UCP/CNRS-UMR8051  
Cergy-Pontoise, France  
marina.ivashkina@ensea.fr

Iryna Andriyanova  
ETIS group  
ENSEA/UCP/CNRS-UMR8051  
Cergy-Pontoise, France  
iryna.andriyanova@ensea.fr

Pablo Piantanida  
Dept. of Telecoms  
SUPELEC  
Gif-sur-Yvette, France  
pablo.piantanida@supelec.fr

Charly Poulliat  
Dept. of Telecoms  
ENSEEIHT, Toulouse, France  
poulliat@enseeiht.fr

Abstract—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. In this paper, the asymptotic performance of backward and joint iterative decoders over the block-Markov (BM) structure based on sparse-graph codes is investigated. We show that the BM structures based on good error-correcting codes or good error-detecting codes have a different behavior, although both of them can be used to come close to the theoretic limit of the binary erasure relay channel (BE-RC).

I. INTRODUCTION

Over the last decade, coding for cooperative communications using the Decode-and-Forward strategy has received a growing attention from both the coding theory and communication theory communities. Since the early works on distributed turbo-codes [1], they have been numerous coding schemes that have been proposed in different contexts, i.e. considering different duplexing mode (half- (HD) vs full-duplex (FD) communications) or addressing communications over different channels (e.g. erasure, Gaussian or Rayleigh-fading channels) [1]-[12]. Among the proposed coding schemes, sparse-graph codes based coding schemes have been mainly considered, and their performances have been mainly evaluated over HD relay channels (in both orthoginal and non-orthoginal multiple accesses to the destination). For sparse-graph based codes, an efficient coding approach has been proposed in [7], that is referred to as bilayer Low-Density Parity-Check (LDPC) codes. This coding approach has given rise to numerous extensions or improvements (see [10] and references therein). Another recent work is [12], where spatially-coupled LDPC codes for orthogonal erasure relay channels were considered.

Recently, we performed the asymptotic analysis of bilayer sparse-graph-based codes for the block-Markov (BM) DF scheme, in both HD and FD regimes [13]. Joint iterative decoding and backward decoding algorithms were considered. We observed in [13] that the asymptotic performance of the block-Markov structure, based on bilayer LDPC codes, was worse than the asymptotic performance of the non-block-Markov scheme. Given that a block-Markov structure is implicitly presumed in the non-orthogonal scenario, does it mean that sparse-graph codes will perform badly in the non-orthogonal regime? The aim of the present paper is to respond to this question and to study in which ways one could attain the theoretical limit in the non-orthogonal scenario. The orthogonal scenario has also been considered.

II. MODEL

In this paper, we consider a full-duplex relay channel that consists of the source $S$, communicating with the destination $D$, with the help of the node $R$. At $R$, the Decode-and-Forward transmission protocol is assumed [14] ($R$ receives the source message, decodes, re-encodes and re-sends it to $D$. We distinguish two reception scenarios: orthogonal (OR), when $D$ receives two messages from $S$ and $R$ separately, and non-orthogonal (NOR), when $D$ receives a mix of messages.

The generation of two messages ($X_1$ at $S$ and $X_2$ at $R$) is based on the block-Markov encoding scheme as given in [13]. Let $w_i$ be the $i$-th information message $i = 1 \ldots M$. At $S$, $w_i$ is encoded using a code $C_S$ into the message $X_1(w_i)$. The message sent by the source is formed as follows

$$X_1(w_i|s_i) = X_1(w_i) \oplus X_2(s_i),$$

where $s_i$ is the bin number corresponding to the information message $w_{i-1}$ and $X_2(s_i)$ is a codeword of a code $C_R$.

This encoding scheme is associated with the following transmission protocol:

- Let $s_1 = 0$.
- For $i$ from 1 to $M$, repeat:
  - $X_1(w_i)$ is generated from $C_S$ at the source end and $X_1(w_i|s_i)$ is broadcasted.
  - $R$ receives $Y_1(i)$, a noisy version of $X_1(w_i|s_i)$, estimates $X_1(w_i)$ and computes $s_i+1$. Then $X_2(s_i+1)$ is generated from $C_R$ and is sent during the next transmission time;
  - at time $i$, in the non-orthogonal scenario, $D$ receives $Y(i)$, based on $X_1(w_i|s_i)$ and on $X_2(s_i)$, while in the orthogonal case, $D$ receives $Y_S$ and $Y_R$, noisy versions of $X_1(w_i|s_i)$ and on $X_2(s_i)$, separately.
- For $i = M + 1$, $R$ sends $X_2(s_M)$ while $S$ broadcasts $w_{M+1}$ set to 0.

For the sake of simplicity, we assume the transmission over the binary erasure relay channel (BE-RC), with erasure probability $\epsilon_{SD}$ on the link SD and erasure probability $\epsilon_{RD}$ on the link RD. In the NOR regime, the channel at D
The transmission protocol above provides the block-Markov (BM) structure depicted in Fig.1. $C_S$ and $C_R$ blocks represent coding schemes at S and at R. This structure can be decoded using numerous decoding schedules. In this paper we consider joint iterative decoder that consists in the following:

- at update 1, 1 < $i$ < $L$, one forward and one backward steps are performed, during which estimates of $X_1^{(i)}(w_i|s_i)$ and $X_2^{(i)}(s_{i+1})$ are calculated, for all 1 ≤ $i$ ≤ $M$, given backward and forward estimations at the previous update 1;
- the final estimate $\hat{w}_i$ is computed for all 1 ≤ $i$ ≤ $M$, given $X_1^{(L)}(w_i|s_i)$ and $X_2^{(L)}(s_{i+1})$.

A backward decoder will be also considered. The backward decoder can be seen as the first backward step of the joint iterative decoder.

Let us define the decoding convergence region:

Definition 1: The decoding (per code block, backward or joint iterative) is assumed to be successful if all corresponding information messages $w_i$, $\forall i \in [1, M]$ have been successfully recovered. Hence the convergence region of the corresponding decoder is the set of all input parameters such that the decoding is successful.

In what follows, the asymptotic decoding performance of backward and joint iterative decoders over BM structures is investigated.

III. FORWARD/BACKWARD DECODING UPDATES OVER THE BM STRUCTURE

The BM structure can be seen as a sequence of alternating blocks $C_S-C_R$ ("coding blocks"), representing a joint coding scheme at $S$ and at $R$, and $T$ ("channel block"), representing the channel function. An example of this representation for the OR scenario is given in Fig.2. In this section we are going to describe forward/backward updates over the BM structure, that make part of backward and joint iterative algorithms.

A. $C_S-C_R$ Block and Update Functions $F_{\beta}^{\text{fwd}}, F_{\alpha}^{\text{fwd}}$

First, let us consider a block $C_S-C_R$.

Definition 2: For a code block $i$, let $\alpha_{i-1}/\beta_i$ and $\beta_{i-1}/\beta_i$ be average forward and backward input/output erasure probabilities, corresponding to $X_1(w_i)/X_2(s_i)$, as it is shown in Fig.3. Hence we define the forward/backward update function:

Definition 3: At the forward/backward update 1 for block $i$, the forward/backward code functions $F_{\beta}^{\text{fwd}}, F_{\alpha}^{\text{fwd}}$, parametrized by $\beta_i$, are defined as follows:

$$F_{\beta}^{\text{fwd}} : \alpha_i(1) = F_{\beta}^{\text{fwd}}(\alpha_{i-1})$$ (2)

$$F_{\alpha}^{\text{fwd}} : \beta_{i-1}(1) = F_{\alpha}^{\text{fwd}}(\beta_i(1))$$ (3)

Definition 4: Let $t_i$ denote the decoding erasure probability at information positions of the code block $i$. Then we define the information update function $F^{\text{inf}}(\alpha_i, \beta_i)$ so that $t_i = F_{\alpha}^{\text{inf}}(\alpha_i, \beta_i)$.

Proposition 1: It can be shown that the following holds:

- 1) $F_x(y)$ is non-decreasing both with $x$ with $y$;
- 2) $F_0(0) = 0$ and $F_1(1) = 1$;
- 3) $F_{\beta}^{\text{fwd}}(a) \geq F^{\text{inf}}(a, \text{bwd})$ and $F_{\beta}^{\text{bwd}}(a) \geq F^{\text{inf}}(a, \text{bwd})$.

Notation 1: Let us denote the convergence region of a $C_S-C_R$ block by $\Gamma$, and the boundary of this region by $\gamma$. Note that one can define boundaries related to one forward/backward update as follows:

$$\gamma^{\text{fwd}}(\beta) = \max_{0 \geq \alpha \geq 1} \{\alpha : F_{\beta}^{\text{fwd}}(\alpha) = 0\}$$

$$\gamma^{\text{bwd}}(\alpha) = \max_{0 \geq \beta \geq 1} \{\beta : F_{\alpha}^{\text{bwd}}(\beta) = 0\}$$

B. $T$ block and Update Functions $G_{\beta}^{\text{fwd}}, G_{\alpha}^{\text{fwd}}$

Let erasure probabilities of S-D and R-D links be $\epsilon_{SD}$ and $\epsilon_{RD}$ respectively.

Definition 5: At update 1, we define forward/backward channel functions as follows (see Fig.4 for illustration):

$$G_{\epsilon_{SD}, \epsilon_{RD}}^{\text{fwd}}(\alpha) = G_{\epsilon_{SD}, \epsilon_{RD}}^{\text{fwd}}(\hat{\alpha}),$$ (4)

$$G_{\epsilon_{SD}, \epsilon_{RD}}^{\text{bwd}}(\beta) = G_{\epsilon_{SD}, \epsilon_{RD}}^{\text{bwd}}(\hat{\beta}).$$ (5)
In what follows, the subscript \((\epsilon_{SD}, \epsilon_{RD})\) is omitted for simplicity.

Note that, if for the BE-RC case, \(G^{fwd}(y)\) and \(G^{bwd}(y)\) are:

1) Orthogonal reception (OR) scenario:

\[
G^{fwd}(y) = 1 - (1 - \epsilon_{SD})(1 - \epsilon_{RD}) \quad (6)
\]
\[
G^{bwd}(y) = \epsilon_{RD}(1 - (1 - \epsilon_{SD})(1 - y)) \quad (7)
\]

2) Non-orthogonal reception (NOR) scenario:

\[
G^{fwd}(y) = \epsilon_{SD}\epsilon_{RD} + \epsilon_{SD}\epsilon_{RD} + \epsilon_{RD}\epsilon_{SD}y \quad (8)
\]
\[
G^{bwd}(y) = \epsilon_{SD}\epsilon_{RD} + \epsilon_{SD}\epsilon_{RD} + \epsilon_{SD}\epsilon_{RD}y \quad (9)
\]

C. Expressions of Forward/Backward Updates for the Joint Iterative Decoder

Let us consider some interesting classes of forward functions defined above. Recall that the first message of \(R\) and the last message of \(S\) are known at \(D\), so initial forward/backward erasure probabilities are:

\[
\alpha_i^{(0)} = \begin{cases} 
G^{fwd}(1), & \text{if } 2 \leq i \leq M, \\
G^{fwd}(0), & \text{if } i = 1,
\end{cases} \quad (10)
\]
\[
\beta_i^{(0)} = \begin{cases} 
G^{bwd}(1), & \text{if } 1 \leq i \leq M - 1, \\
G^{bwd}(0), & \text{if } i = M,
\end{cases} \quad (11)
\]

and the forward/backward update \(1 > 0\) can be written as:

\[
\alpha_i^{(1)} = G^{fwd}\left(F^{\beta_i^{(1-1)}(\alpha_i^{(1-1)})}\right) \quad (12)
\]
\[
\beta_i^{(1)} = G^{bwd}\left(F^{\beta_i^{(1-1)}(\beta_i^{(1-1)})}\right) \quad (13)
\]

Remark 1: The density evolution presented above can be represented graphically (Fig.5), similar to EXIT charts [15].

Notation 2: Denote by \(\Gamma^{BMD}\) and \(\Gamma^{JNT}\) the convergence regions of backward and joint iterative decoders.

D. Two Classes of Codes: Erasure-Correcting (EC) Codes and Erasure-Detecting (ED) Codes

Let us consider some interesting classes of forward functions\(^3\) \(F^{fwd}\). Consider the asymptotic iterative performance of a \(C_S - C_R\) block. Then we consider two cases:

\(a\) \(F^{fwd} = \begin{cases} 
\approx 1, & \epsilon^* \leq \epsilon^* \leq 1 \\
0, & \epsilon < \epsilon^*
\end{cases} \)

\(^3\)The backward decoder is a particular case of the joint iterative one.

\(^2\)Or, similarly, of \(F^{bwd}\).

(b) \(F^{fwd}\) is strictly increasing with \(\epsilon, \epsilon \in [0, 1]\).

Note that \(F\) functions of basically all efficient erasure-correcting codes (e.g. LDPC, turbo codes) belong to the class \((a)\). The class \((b)\) represents the class of erasure-detecting codes, which are typically bad erasure-correctors (e.g. LDGM). Hence let us denote these classes as Erasure-Correcting class of codes (EC) and Erasure-Detecting class of codes (ED), and evaluate their performance in the block-Markov encoding structure.

IV. BLOCK-MARKOV STRUCTURE BASED ON EC CODES

We have the following important lemma:

**Lemma 1:** For a BM structure based on EC codes,

\[
\Gamma^{BMD} = \Gamma(G^{fwd}(1), G^{bwd}(0)), \quad \Gamma^{JNT} = \Gamma(G^{fwd}(1), G^{bwd}(0)) \cup \Gamma(G^{fwd}(0), G^{bwd}(1))
\]

**Proof:** Consider the forward update (the backward case is similar). Note that, for EC codes:

\[
F^{fwd}_{\alpha}(\beta) \approx 1(\alpha > \gamma^{fwd}(\beta)), \quad F^{fwd}_{\beta}(\alpha) \approx 1(\beta > \gamma^{fwd}(\alpha)).
\]

Consider the case when \(F^{fwd}_{\alpha}(\alpha_0^{(1)}) = F^{fwd}_{\beta}(\alpha_0^{(1)})(G^{fwd}(0)) = 0\) (bits of \(X_2(s_1)\) are decoded perfectly). Then \(\alpha_0^{(1)} = G^{fwd}(0)\), implying \(F^{fwd}_{\alpha}(\alpha_0^{(1)}) = 0\), for \(i = 1 \ldots M\). As \(F^{fwd}_{\alpha}(\alpha_1^{(1)}) \leq F^{fwd}_{\beta}(\alpha_0^{(1)}) = 0\), the decoder converges in one forward update.

Now let \(G^{fwd}(0) > \gamma^{fwd}(G^{bwd}(1))\) (bits of \(X_2(s_1)\) remained completely unknown and \(\alpha_0^{(1)} = G^{fwd}(1)\)). By non-decreasing property of both \(G^{fwd}\) and \(F^{fwd}\) we get that \(F^{fwd}_{\beta}(\alpha_1^{(1)}) = 1\), for \(i = 1 \ldots M\) and decoding fails.

The theorem below follows directly Lemma 1.

**Theorem 1:** Convergence regions of the BM structure based on EC codes is bounded by the convergence region of one \(C_S - C_R\) block, i.e.

\[\Gamma^{BMD} \subseteq \Gamma \quad \text{and} \quad \Gamma^{JNT} \subseteq \Gamma.\]

Thus in order to design a capacity-approaching BM structure based on EC codes, one should necessarily choose a capacity-approaching \(C_S - C_R\) code.
V. Block-Markov Structure Based on ED Codes

Similarly to Lemma 1, for ED codes:

**Lemma 2:** Assuming that the BM decoding process converges sufficiently fast, threshold boundaries of the BM structure based on an ED ensemble are given by:

\[ 
\Gamma_{\text{EMD}} = \Gamma(G^{\text{fwd}}(1), G^{\text{bwd}}(0)) \tag{14} 
\]

\[ 
\Gamma_{\text{JNT}} = \Gamma(\alpha_{\infty}^{(i)}, \beta_{\infty}^{(i)}) \tag{15} 
\]

The proof follows directly from definition of \( F^{\text{fwd}} \) and \( F^{\text{bwd}} \) for ED codes.

From Lemma 2 the following result follows directly:

**Theorem 2:** Convergence regions of the BM structure based on ED codes have the following property:

\[ 
\Gamma_{\text{EMD}} \supseteq \Gamma \quad \text{and} \quad \Gamma_{\text{JNT}} \supseteq \Gamma. 
\]

We obtain an interesting result here: the iterative performance of the BM structure based on ED codes may outperform the performance of one erasure-detecting \( C_S - C_R \) block! The question is whether it can attain the theoretical limit for any value of \( \epsilon_{\text{SD}} \)? Unfortunately, the answer is no. To show it, let us consider two extreme points of the threshold boundary for the BM structure based on ED codes: \( \epsilon_{\text{SD}} = 1 \) and \( \epsilon_{\text{RD}} = 1 \).

**Theorem 3:** The block-Markov structure based on ED codes is bounded away from capacity for \( \epsilon_{\text{SD}} = 1 \).

**Proof:** Without loss of generality, let us consider the orthogonal (OR) scenario. For \( \epsilon_{\text{SD}} = 1 \), channel functions \( G^{\text{fwd}} \) and \( G^{\text{bwd}} \) can be written as \( G^{\text{fwd}}(y) = 1 \) and \( G^{\text{bwd}}(y) = \epsilon_{\text{RD}} \), implying that the BM structure is broken into separate blocks \( C_S - C_R \). Therefore, the iterative threshold of the BM structure is equal to the one of the block \( C_S - C_R \).

Note that a different thing happens in the region \( \epsilon_{\text{RD}} \) close to 1: in this case, channel functions are given by \( G^{\text{bwd}}(y) = G^{\text{fwd}}(y) = 1 - \epsilon_{\text{SD}} \), and the BM structure may be capacity-approaching.

A. Example of a BM Ensemble Based on ED Codes

To illustrate our analysis, let us give an example of a BM code, based on LDGM codes:

**Definition 6:** Bilayer Lengthened LDGM (BL-LDGM) codes are bilayer \( C_S - C_R \) codes such that \( C_S \) and \( C_R \) are two LDGM codes, sharing their information bits and having respective degree distributions \((\lambda_S(x), \rho_S(x))\) and \((\lambda_R(x), \rho_R(x))\).

**Notation 3:** Let us denote by BM-BL-LDGM the BM structure based on bilayer LDGM codes.

Let us optimize distributions \( \lambda_S(x) \) and \( \lambda_R(x) \) (for simplicity, a constant check node degree is assumed). We define the following optimization problem that maximizes the rate of the BL-LDGM code:

\[ 
\max \sum_{i=1}^{i_{\text{max}}} \frac{\lambda_{S,i}}{i}, \tag{16} 
\]

with \( (\alpha_{\infty}^{(i)}, \beta_{\infty}^{(i)}) \in \Gamma \)

\[ 
\sum_{i=1}^{i_{\text{max}}} \lambda_{S,i} = 1 \quad \text{and} \quad \lambda_{S,l} \geq 0, \quad \forall 1 \leq l \leq i_{\text{max}}. 
\]

This is a non-linear optimization problem, and it can be solved numerically.

**Example 1:** We start by fixing \( C_R \) to be a regular \((x^4, x^4)\) LDGM ensemble. It has an iterative decoding threshold \( \epsilon^* = 0.313 \). We are going to optimize \( \lambda_R(x) \) by the optimization algorithm stated above, using the blind random search optimization [16]. The obtained degree distribution is

\[ 
\lambda_S(x) = 0.0481x + 0.048x^2 + 0.0199x^3 + 0.0196x^4 + 0.046x^5 + 0.0441x^6 + 0.1029x^7 + 0.0639x^8 + 0.061x^9 + 0.0172x^{10} + 0.1163x^{11} + 0.0885x^{12} + 0.1096x^{13} + 0.0861x^{14} + 0.1291x^{15} 
\]

The threshold boundary of the BM structure with the optimized \( C_R \) part is shown in Fig. 6: dashed lines represent the orthogonal reception (OR), and full lines represent the non-orthogonal (NOR) one. Black curves correspond to threshold boundaries; achievable rates region are given by grey lines. Note that for \( \epsilon_{\text{RD}} = 1 \), the BM-BL-LDGM ensemble has multiplicative gap to capacity \( \delta \approx 0.003 \), due to the optimization of the degree distribution. For \( \epsilon_{\text{SD}} = 1 \), the gap to capacity is around 0.35 (the extreme point corresponds to the iterative threshold of the LDGM serving as \( C_R \)). Note that obtained results are consistent with our analysis.

VI. OUR NEW CONSTRUCTION

We would like to design a new \( C_S - C_R \) code family which would be capacity-approaching at two points \( \epsilon_{\text{SD}} = 1 \) and \( \epsilon_{\text{RD}} = 1 \) simultaneously, both in orthogonal and non-orthogonal scenarios. We propose the following code structure:

**Definition 7:** Bilayer-Expurged LDGM (BE-LDGM) codes are bilayer \( C_S - C_R \) codes such that \( C_S \) and \( C_R \) codes are given respectively by an LDGM and an LDPC codes, that share their information bits. Corresponding degree distributions are denoted by \((\lambda_S(x), \rho_S(x))\) and \((\lambda_R(x), \rho_R(x))\).

We develop the following design condition:

**Proposition 2:** Consider a BE-LDGM ensemble, and let \( \epsilon_{\text{LDPC}} \) be the iterative threshold of the constituent LDPC part. The BE-LDGM ensemble simultaneously approaches the
shown that for EC codes, the BM performance is limited the performance of one block in the BM structure. For ED codes, the BM performance may outperform the performance of one block. It has been also shown that the performance of a BM-ED code cannot be improved over $\epsilon_R$ and $\epsilon_S$ simultaneously. A new construction for the $C_S - C_R$ block to overcome this limitation has been proposed. The new BM schemes based on the new family perform close to the theoretical limit. The question whether the new codes can be capacity-approaching is still open.

REFERENCES


VII. ACHIEVEMENT

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VIII. CONCLUSION

In this paper, we have considered asymptotic convergence regions of the block-Markov structure for the binary erasure relay channel, based on EC or ED codes, both for orthogonal and non-orthogonal reception at the destination. It has been