New Upper Bounds on the Expected Length of One-to-One Codes

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Abstract — **We present a new lower bound on differential entropy and use it to tighten Wyner's lower bound on discrete entropy, $H(X) \geq \mathrm{E}\log X$. As an application example, we derive tighter upper bounds on the expected length of binary one-to-one codes.**

I. Lower bounds on entropy

The logarithm of the geometric mean of a continuous random variable with a monotone probability density yields a lower bound on its differential entropy [2]:

Theorem 1 *Let X be a finite variance real random variable with a monotone one-sided pdf f and range $[x_0, \infty)$ or $(-\infty, x_0]$. Then $h(X) \geq \mathrm{E}\log|X - x_0| + \log e$ with equality iff f is a uniform density.*

This result can be used to obtain an analogous bound on discrete entropy. Any countable (discrete) source alphabet $\mathcal{X}$ can be indexed by the natural numbers such that the associated probabilities are monotone decreasing with the index. Hence in the following we assume $\mathcal{X} = \{1, 2, \ldots, N\}$, where N can be infinite, and that the probabilities $p_i = \Pr\{X = i\}$ are monotone decreasing: $p_1 \geq p_2 \geq \ldots \geq p_N$. We obtain a bound on the entropy of such *monotone probability distributions* by applying Theorem 1 to a random variable with a suitably defined piecewise constant pdf.

Theorem 2 *For a positive integer valued random variable X with a monotone probability distribution and finite entropy,*

$$H(X) \geq \sum_{n=1}^{N} \left[p_n \log n + p_{n+1} \cdot n \log \frac{n+1}{n} \right] \tag{1}$$

with equality iff X is uniformly distributed on $\{1, 2, \ldots, N\}$. This can be weakened to

$$H(X) \geq \mathrm{E}\log X + (1 - p_1)\log 2 \tag{2}$$

with equality iff $X \equiv 1$ or X is uniformly distributed on $\{1, 2\}$.

The bounds (1) and (2) are slightly tighter than Wyner's inequality $H(X) \geq \mathrm{E}\log X$ proved in [3].

II. One-to-One Codes

A binary one-to-one encoding of a random variable X with countable alphabet $\mathcal{X}$ is an injection $\phi : \mathcal{X} \to \{0, 1\}^*$, the set of finite binary strings (including the empty string ϵ). As before, we assume $\mathcal{X} = \{1, 2, \ldots, N\}$ and monotone decreasing probabilities. It is easy to show that any encoding that achieves the minimal expected code length $L_\epsilon(X)$ assigns codewords of length $|\phi(n)| = \lfloor \log n \rfloor$ (all logarithms in this section are to the base two). Hence Wyner's inequality is an upper bound on the expected length of one-to-one codes:

$$L_\epsilon(X) = \mathrm{E}\lfloor \log X \rfloor \leq \mathrm{E}\log X \leq H(X).$$

It follows immediately that Theorem 2 leads to tighter upper bounds on $L_\epsilon(X)$. The figure displays the bound (2) as a function of p_1, the largest probability. Wyner's bound corresponds to the (top) horizontal axis. By considering X as a refinement of a binary random variable with distribution $(p_1, 1 - p_1)$ we obtain an even tighter bound:

Theorem 3

$$L_\epsilon(X) \leq \begin{cases} H(X) - h(p_1) + p_1 \log \frac{4k^{k+1}}{(k+1)^k} + \log \frac{k+1}{2k}, \\ \qquad \textit{for } 0 \leq p_1 \leq \frac{1}{k},\ k = 2, 3, \ldots \\ H(X) - h(p_1) + 1 - p_1, \quad \textit{for } \frac{1}{2} \leq p_1 \leq 1 \end{cases} \tag{3}$$

As can be seen in the figure, for $p_1 \geq \frac{1}{2}$ the bound (3) coincides with Blundo and De Prisco's bound, which is shown for comparison (equation (7) in [1]). Notice that the derivative of (3) is discontinuous at the points $p_1 = \frac{1}{k}$, $k = 2, 3, \ldots$.

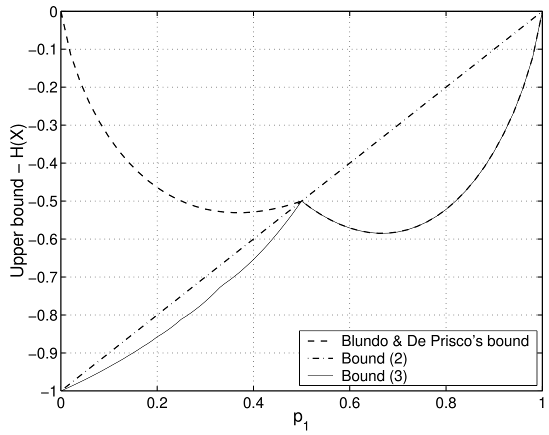

Figure 1: Upper bounds on the expected code length of binary one-to-one encodings.

[1] This work was mainly done while the author was with the Communication Systems Department, EPFL, Lausanne, Switzerland.

References

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[3] A. Wyner, "An upper bound on the entropy series", *Inform. Contr.*, vol. 20, pp. 176–181, 1972.