Rate Distortion Bounds via Threshold-based Classification

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Abstract — **We present parametric upper bounds on the distortion rate function of memoryless sources. In particular, these bounds are useful for sparse sources with a peaked, heavy-tailed pdf. In addition we study the geometric mean as a measure of sparseness.**

I. Introduction

We consider lossy encoding of a real-valued memoryless source using the squared error distortion measure. Further we assume that the source has a symmetric pdf, $f(x) = f(-x)$, noting that the results can be easily generalized.

Each source sample is classified by comparing its magnitude with a threshold T: the samples above threshold ($|x_i| \geq T$) are called *significant*, the others ($|x_i| < T$), *insignificant*. The significant samples are characterized by their ratio $\mu(T) = \Pr\{|X| \geq T\} = 2\int_T^\infty f(x)\,dx$ and their unnormalized variance $A(T) = \mu(T)\,\mathrm{E}[X^2|\,|X| \geq T] = 2\int_T^\infty f(x)x^2\,\mathrm{d}x$, where $A(0) = \sigma^2$ is the source variance. The classification decision is sent as side information to the decoder, using $h_b(\mu)$ nats per sample.

II. Threshold-based Bounds

The distortion rate function (drf) of the significant samples can be upper bounded by the drf of a Gaussian with the same variance. We add the distortion from the insignificant samples (which are implicitly quantized to zero) to get a low-rate upper bound on the drf of the original source, given a threshold $t \geq 0$ and rate $R \geq h_b(\mu(t))$:

$$D(R) \leq A(t)\exp\left(-2\frac{R - h_b(\mu(t))}{\mu(t)}\right) + \sigma^2 - A(t). \qquad (1)$$

We obtain a parametric bound by determining the rate $R^*(t)$ at which the bound (1) is tighter than the bounds for thresholds in a neighborhood of t. An exact expression for $R^*(t)$ was first presented in [1] (detailed proof in [2]); here we give an approximation that yields a slightly weaker bound, but is more insightful:

$$R^*(t) \approx R_2(t) = h_b(\mu(t)) + \tfrac{\mu(t)}{2}\log\left(-2\gamma^{-1}(t)\log\mu(t)\right), \qquad (2)$$

where $\gamma^{-1}(t)$ is the normalized tail variance

$$\gamma^{-1}(t) = \tfrac{A(t)}{\mu(t)\,t^2} = \mathrm{E}[X^2|\,X^2 \geq t^2]/t^2. \qquad (3)$$

The resulting low-rate upper bound is:

$$D(R_2(t)) \leq \tfrac{A(t)}{-2\gamma^{-1}(t)\log\mu(t)} + \sigma^2 - A(t). \qquad (4)$$

If we apply the Gaussian upper bound also to the insignificant samples, we obtain a high-rate bound via standard waterfilling techniques. Let the variances of the insignificant and the significant samples be $\sigma_0^2(t) = \frac{\sigma^2 - A(t)}{1-\mu(t)}$ and $\sigma_1^2(t) = \frac{A(t)}{\mu(t)}$, respectively. Then for all $R \geq h_b(\mu(t)) + \frac{1}{2}\log\frac{\sigma_1^2(t)}{\sigma_0^2(t)}$ we have

$$D(R) \leq c(t)\,\sigma^2 e^{-2R}, \qquad (5)$$

where

$$c(t) = \exp\left(2h_b(\mu(t)) + (1-\mu(t))\log\tfrac{\sigma_0^2}{\sigma^2} + \mu(t)\log\tfrac{\sigma_1^2}{\sigma^2}\right). \qquad (6)$$

The tightest such bound is obtained by numerically searching the $t_0 \in [0,\infty)$ minimizing $c(t)$. Since $\lim_{t\to+0} c(t) = 1$, the Gaussian upper bound is always a member of this family. We remark that (5) can also be turned into an upper bound on differential entropy.

III. Sparse Sources

By construction, the above bounds are well matched to sources with peaked, heavy-tailed pdf, that is *sparse* sources. However, they require (an estimate of) the source pdf and are thus inadequate for a simple characterization of source sparseness. We argue that the logarithm of the geometric mean, $\mathrm{E}\log|X|$, is a good measure of sparseness. In combination with the variance, the geometric mean allows one to bound the source entropy and thus characterize the high-rate $R(D)$ behavior of sparse sources.

Via the calculus of variations we find the maximum entropy pdf given the constraints $\mathrm{E}\,X^2 = \sigma^2$ and $\mathrm{E}\log|X| = \theta$:

$$f(x) = \Gamma^{-1}(\tfrac{u}{2})\left(\tfrac{u}{2\sigma^2}\right)^{u/2}|x|^{u-1}\exp\left(-\tfrac{ux^2}{2\sigma^2}\right), \qquad (7)$$

where u is the unique solution to $\theta = \frac{1}{2}\Psi(\frac{u}{2}) - \frac{1}{2}\log\frac{u}{2\sigma^2}$. Thus we have an upper bound on the entropy of all continuous pdf's satisfying the variance and geometric mean constraints:

$$h(X) \leq \tfrac{u}{2} - \tfrac{u-1}{2}\Psi(\tfrac{u}{2}) + \log\Gamma(\tfrac{u}{2}) - \tfrac{1}{2}\log\tfrac{u}{2\sigma^2}. \qquad (8)$$

If the source pdf $f(x)$ is symmetric and unimodal (decreasing in $|x|$) the following lower bound holds:

$$h(X) \geq \mathrm{E}\log|X| + \log 2. \qquad (9)$$

This can be proved using Riemann integration and Wyner's inequality [4], i.e. the discrete entropy counterpart of this bound.

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References

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