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# Wavelets and curvelets for image deconvolution: a combined approach

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**Abstract**

We propose in this paper a new deconvolution approach, which uses both the wavelet transform and the curvelet transform in order to benefit from the advantages of each. We illustrate the results with simulations.


*Keywords:* Wavelet; Curvelet; Filtering; Deconvolution

## 1. Introduction

It has been shown [11] that, for denoising problems, the curvelet transform approach outputs a PSNR comparable to that obtained via the undecimated wavelet transform, but the curvelet reconstruction does not contain as many disturbing artifacts along edges that one sees in wavelet reconstructions. Although the results obtained by simply thresholding the curvelet expansion are encouraging, there is of course ample room for further improvement. A quick inspection of the residual images resulting from the *Lena* image filtering (a $3\sigma$ hard thresholding has been applied with both transforms) for both the wavelet and curvelet transforms shown in Fig. 1 reveals the presence of very different features. For instance, wavelets do not restore long edges with high fidelity while curvelets are challenged by small features such as *Lena*'s eyes. Loosely speaking, each transform has its own area of expertise and this complementarity may be of great potential.

In [12], a denoising algorithm was proposed which investigates this complementarity, by combining several multiscale transforms in order to achieve very high quality image restoration. For numerical reasons, the choice is restricted to the transforms which have a fast forward and inverse implementation. Considering $K$ linear transforms $\mathcal{T}_1,\ldots,\mathcal{T}_K$ (respectively $\mathcal{R}_1,\ldots,\mathcal{R}_K$ the inverse transforms, and we have $\mathcal{R}_k = \mathcal{T}_k^{-1}$ for an orthogonal transform), the combined filtering method (CFM) consists of minimizing a functional such as the Total Variation (TV) or the $l_1$ norm of the multiscale coefficients, but under a set of constraints in the transform domains. Such

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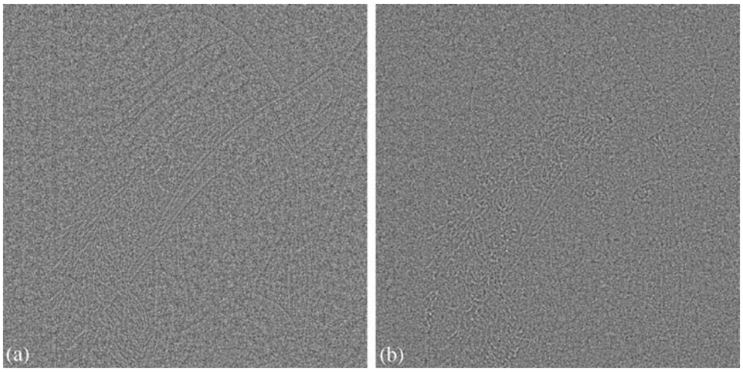

Fig. 1. Residual following thresholding of the undecimated wavelet transform (left) and thresholding of the curvelet transform (right).

constraints express the idea that if a significant coefficient is detected by a given transform $\mathcal{T}_k$ at a scale $j$ and at a pixel index $l$, then the transformation of the solution must reproduce the same coefficient value at the same scale and the same position. In short, the constraints guarantee that the reconstruction will take into account any pattern which is detected as significant by any of the $K$ transforms. Given data $y$ of the form $y = s + \sigma z$, where $s$ is the image to recover and $z$ is standard white noise, the combined filtering method consists of solving the following optimization problem:

$$\min S(\tilde{s}), \quad \text{subject to} \quad s \in C, \tag{1}$$

where $S(\tilde{s})$ can be either an $\ell_1$ penalty on the coefficient (i.e. $S(\tilde{s}) = \sum_k \|\mathcal{T}_k \tilde{s}\|_{\ell_1}$) or the Total Variation norm, and $C$ is the set of vectors $\tilde{s}$ which obey the linear constraints

$$\tilde{s} \geqslant 0,$$

$$|\mathcal{T}_k \tilde{s} - \mathcal{T}_k y| \leqslant e \quad \text{for all } k \tag{2}$$

The second inequality constraint only concerns the set of significant coefficients, i.e. those indices $\mu$ such that $\alpha_\mu = (\mathcal{T}_k y)_\mu$ exceeds (in absolute value) a threshold $t_\mu$. More details can be found in [12].

Several papers have been recently published, based on the concept of minimizing the Total Variation under constraints in the wavelet domain [6,3,8] or in the curvelet domain [2]. CFM [12] can be seen as a generalization of these methods.

Section 2 introduces the deconvolution problem, and discusses different wavelet based methods and Section 3 shows how a deconvolution can be derived from a combined approach.

## 2. Wavelets and deconvolution

Consider an image characterized by its intensity distribution $I$, corresponding to the observation of a "real image" $O$ through an optical system. If the imaging system is linear and shift-invariant, the relation between the data and the image in the same coordinate frame is a convolution: $I(x,y) = (P*O)(x,y) + N(x,y)$, where $P$ is the point spread function (PSF) of the imaging system, and $N$ is additive noise. We want to determine $O(x,y)$ knowing $I$ and $P$. This inverse problem has led to a large amount of work, the main difficulties being the existence of: (i) a cut-off frequency of the PSF, and (ii) the additive noise (see for example [1]).

The wavelet based non-iterative algorithm, the wavelet-vaguelette decomposition [5], consists of first applying an inverse filtering ($F = P^{-1} * I = O + P^{-1} * N = O + Z$ where $\hat{P}^{-1}(v) = 1/\hat{P}(v)$). The noise $Z = P^{-1} * N$ is not white but remains Gaussian. It is amplified when the deconvolution problem is unstable. Then, a wavelet transform is applied on $F$, the wavelet coefficients are soft or hard thresholded [4], and the inverse wavelet transform furnishes the solution.

The method has been refined by adapting the wavelet basis to the frequency response of the inverse of $P$ [7]. This leads to a special basis, the *Mirror Wavelet Basis*. This basis has a time-frequency tiling structure different from the conventional wavelets one. It isolates the frequency $\nu_s$ where $\hat{P}$ is close to zero, because a singularity in $\hat{P}^{-1}(\nu_s)$ influences the noise variance in the wavelet scale corresponding to the frequency band which includes $\nu_s$. Because it may not be possible to isolate all singularities, Neelamani [9] has advocated a hybrid approach, and proposes to still use the Fourier domain to restrict excessive noise amplification. These approaches are fast and competitive compared to linear methods, and the wavelet thresholding removes the Gibbs oscillations. This presents however several drawbacks: (i) the first step (division in the Fourier space by the PSF) cannot always be done properly (for example when the frequency cut-off $\nu_c$ is smaller than the Nyquist frequency, then $\hat{P}(\nu)$ equals zero for all $\nu > \nu_c$), (ii) the positivity prior is not used, and (iii) it is not trivial to consider non-Gaussian noise.

As an alternative, several wavelet-based iterative algorithms have been proposed [13], especially in the astronomical domain where the positivity prior is known to improve significantly the result. The simplest method consists of first estimating the multiresolution support $M$ (i.e. $M(j,l) = 1$ if the wavelet transform of the data presents a significant coefficient at band $j$ and at pixel index $l$, and 0 otherwise) [10], and to apply the following iterative scheme:

$$O^{n+1} = O^n + P^* * \mathcal{R}[M.\mathcal{W}(I - P * O^n)] \tag{3}$$

where $P^*$ is the transpose of the PSF ($P^*(x,y) = P(-x,-y)$), $\mathcal{W}$ is the wavelet transform operator and $\mathcal{R}$ is the wavelet reconstruction operator. At each iteration, information is extracted from the residual only at scales and positions defined by the multiresolution support. $M$ is estimated from the input data and the correct noise modeling can easily be considered [10]. The positivity is introduced in the following way:

$$O^{n+1} = \mathcal{P}_c[O^n + P^* * \mathcal{R}[M.\mathcal{W}(I - P * O^n)]], \tag{4}$$

where $\mathcal{P}_c$ is the projection operator which enforces the positivity (i.e. set to 0 all negative values).

## 3. The combined deconvolution method

Similar to the filtering, we expect that the combination of different transforms can improve the quality of the result. The combined approach for the deconvolution leads to two different methods.

If the noise is Gaussian and if the division by the PSF in the Fourier space can be carried out properly, then the deconvolution problem becomes a filtering problem where the noise is still Gaussian, but not white. The combined filtering Algorithm can then be applied using the curvelet transform and the wavelet transform, but by estimating first the correct thresholds in the different bands of both transforms. Since in many cases the mirror wavelet basis may produce better results than the wavelet basis, it is recommended to use it instead of the standard undecimated wavelet transform.

An iterative deconvolution method is more general and can always be applied. Furthermore, the correct noise modeling can much more easily be taken into account. This approach consists of detecting, first, all the significant coefficients with all multiscale transforms used. If we use $K$ transforms $\mathcal{T}_1, \ldots, \mathcal{T}_K$, we derive $K$ multiresolution supports $M_1, \ldots, M_K$ from the input image $I$ using noise modeling.

For instance, in the case of Poisson noise, we apply the Anscombe transform to the data (i.e. $\mathcal{A}(I) = 2\sqrt{I + \frac{3}{8}}$). Then we detect the significant coefficients with the $k$th transform $\mathcal{T}_k$, assuming Gaussian noise with standard deviation equal to 1, in $\mathcal{T}_k \mathcal{A}(I)$ instead of $\mathcal{T}_k I$. $M_k(j,l) = 1$ if a coefficient in band $j$ at pixel index $l$ is detected, and $M_k(j,l) = 0$ otherwise. For the band $J$ which corresponds to the smooth array (i.e. coarsest resolution) in transforms such as the wavelet or the curvelet transform, we force $M_k(J,l)=1$ for all $l$.

Following determination of a set of multiresolution supports, we propose to solve the following optimization problem:

$$\min_{\tilde{O}} \mathrm{TV}(\tilde{O}),$$

$$\text{subject to} \quad M_k \mathcal{T}_k [P * \tilde{O}] = M_k \mathcal{T}_k I \quad \text{for all } k, \tag{5}$$

where TV is the total-variation, i.e. an edge preservation penalization term defined by:

$$\mathrm{TV}(\tilde{O}) = \int |\nabla \tilde{O}|^p,$$

with $p = 1.1$. We chose $p = 1.1$ in order to approach the case of $p = 1$ with a strictly convex functional.

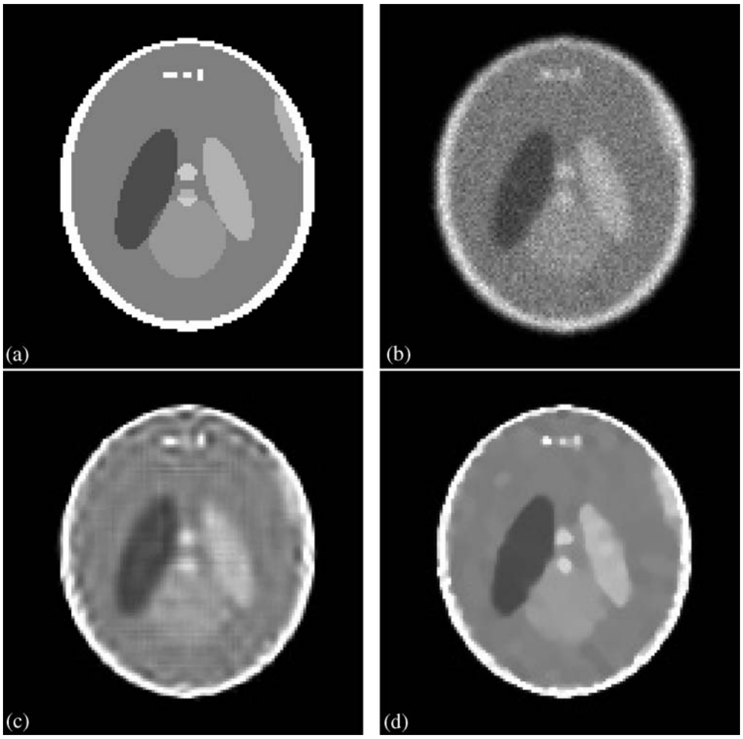


Fig. 2. Top, original image (phantom) and simulated data (i.e. convolved image plus Poisson noise). Bottom, deconvolved image by the wavelet based method and the combined approach.

Minimizing with TV, we force the solution to be closer to a piecewise smooth image.

The constraint imposes fidelity on the data, or more exactly, on the significant coefficients of the data, obtained by the different transforms. Non-significant (i.e. noisy) coefficients are not taken into account, preventing any noise amplification in the final algorithm.

A solution for this problem could be obtained by relaxing the constraint to become an approximate one:

$$\min_{\tilde{O}} \sum_k \|M_k \mathscr{T}_k I - M_k \mathscr{T}_k [P * \tilde{O}]\|_2 + \lambda \mathrm{TV}(\tilde{O}). \quad (6)$$

The solution is computed by using the projected Landweber method [1]:

$$\tilde{O}^{n+1} = \mathscr{P}_c \left[ \tilde{O}^n + \alpha \left( P^* * \bar{R}^n - \lambda \frac{\partial \mathrm{TV}}{\partial O}(\tilde{O}^n) \right) \right], \quad (7)$$

where $\bar{R}^n$ is the significant residual which is obtained using the following algorithm:

- Set $I_0^n = I^n = P * \tilde{O}^n$.
- For $k = 1, \ldots, K$ do $I_k^n = I_{k-1}^n + \mathscr{R}_k [M_k(\mathscr{T}_k I - \mathscr{T}_k I_{k-1}^n)]$
- The significant residual $\bar{R}^n$ is obtained by: $\bar{R}^n = I_K^n - I^n$.

This can be interpreted as a generalization of the multiresolution support constraint to the case where several transforms are used. The order in which the transforms are applied has no effect on the solution. We extract in the residual the information at scales and pixel indices where significant coefficients have been detected.

$\alpha$ is a convergence parameter, chosen either by a line-search minimizing the overall penalty function or as a fixed step-size of moderate value that guarantees convergence, and $\lambda$ is the regularization hyperparameter. Since the noise is controlled by the multiscale transforms, the regularization parameter does not have the same importance as in standard deconvolution methods. A much lower value is enough to remove the artifacts relative to the use of the wavelets and the curvelets. The positivity constraint can be applied at each iteration.

Fig. 2, top, shows the Logan–Shepp Phantom and the simulated data, i.e. original image convolved by a Gaussian PSF (full width at half maximum, FWHM = 3.2) and Poisson noise. Fig. 2, bottom, shows the deconvolution with (left) a pure wavelet deconvolution method (no penalization term) and (right) the combined deconvolution method (parameter $\lambda = 0.4$).

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