ON A CLASS OF GENERALIZED RADON TRANSFORMS AND ITS APPLICATION IN IMAGING SCIENCE

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Abstract. Integral transforms based on geometrical objects, i.e. the so-called generalized Radon transforms, play a key role in integral geometry in the sense of I M Gelfand. In this work, we discuss the properties of a newly established class of Conical Radon Transforms (CRT), which are defined on sets of circular cones having fixed axis direction and variable opening angle. In particular, we describe its inversion process, i.e. the recovery of an unknown function from the set of its integrals on cone surfaces, or its *conical projections*. This transform is the basis for a new gamma-ray emission imaging principle, which works with Compton scattered radiation and offers the remarkable advantage of functioning with a fixed detector instead of a rotating one, as in conventional emission imaging modalities.

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1. Introduction

Integral geometry is the branch of geometrical analysis which treats integral transforms of *geometrical nature*. Through Radon transforms and the like, it provides an elegant way for constructing group representations [1]. But over the decades, integral geometry has also become the mainstay of the mathematics of imaging science and an efficient method for solving inverse problems. The aim in imaging science is to obtain a 3D-image of an object (i.e. its hidden part) from external measurements, without destroying it or without taking it apart. For inverse problems, the objective is to reconstruct a potential from scattering data. Integral geometry has contributed to three Nobel prize works, one of which was awarded in 1979 to A.M. Cormack, a nuclear physicist turned mathematician, for his contributions to computed tomography (CT). The nature and properties of these transforms, in particular their *invertibility*, constitute the main themes of research in this field.

In this work, we discuss a new integral transform, the Conical Radon Transform (CRT), which has not yet been discussed in the literature and which lends support to a new principle of gamma-ray emission imaging. To convey some key ideas, we review the classical Radon transform in two dimensions and its role in imaging science. The conical Radon transform is then introduced as a natural consequence of the Compton scattering of radiation with matter. New inversion formulas are derived by Fourier transform and back-projection approach.

2. The classical Radon Transform (RT): a review

2.1. **Definition.** This transform $\mathcal{R}$ maps $\mathcal{S}$-functions on $\mathbb{R}^2$ to their integrals on straight lines $\zeta \in \mathbb{R}^2$, defined by a normal unit vector $\mathbf{n} = (\cos\phi, \sin\phi)$ and a distance p to the coordinate origin.

Definition 1. Let $\mathbf{n}^\perp$ be the unit vector orthogonal to $\mathbf{n}$. The Radon transform $\mathcal{R}f\,(p,\phi)$ of $f(\mathbf{r})$ is given by the line integral

$$\mathcal{R}f\,(p,\phi) = \int_{-\infty}^{\infty} f(p\mathbf{n} + \sigma\mathbf{n}^\perp)\, d\sigma. \tag{1}$$

$\mathcal{R}f\,(p,\phi)$ represents the gamma-ray activity of a radiating object along the line parameterized by (p,ϕ) and measured on a detector.

Eq. (1) can be rewritten as a Fredholm equation of the first kind in cartesian coordinates, with a δ-function kernel [2], as

$$\mathcal{R}f\,(p,\phi) = \int_{\mathbb{R}^2} dx\,dy\;\delta(p - x\cos\phi - y\sin\phi)\;f(x,y). \tag{2}$$

$\mathcal{R}f$ is related to the 2D-Fourier transform of $f(\mathbf{r})$ by

Theorem 2. *Let $\widetilde{f}(\mathbf{k})$ be the Fourier transform of $f(\mathbf{r})$. Then $\widetilde{\mathcal{R}f}(\nu\mathbf{n}) = \widetilde{f}(\nu\mathbf{n})$. This is known as the Central-Slice-Theorem* [2].

Proof. Using polar coordinates and the Fourier representation of the delta function in eq. (2), we can reexpress $\mathcal{R}f\,(p,\phi)$ with the notations $\nu r\cos(\theta-\phi) = (\nu\mathbf{n}\cdot r\mathbf{k})$ as

$$\mathcal{R}f\,(p,\phi) = \int_{-\infty}^{+\infty} d\nu\, e^{2i\pi\nu p} \int\int_{\mathbb{R}^2} r\, d\theta\, dr\;\, f(r,\theta)\;\, e^{-2i\pi(\nu\mathbf{n}\cdot r\mathbf{k})}. \tag{3}$$

Then it is clear that $\widetilde{\mathcal{R}f}(\nu\mathbf{n}) = \widetilde{f}(\nu\mathbf{n})$, QED. □

2.2. **Inversion.** The *Central Slice Theorem* leads to a first inversion formula by application of the inverse 2D-Fourier transform:

$$f(r,\theta) = -\frac{1}{2\pi^2}\,\text{p.v.}\int_{-\pi}^{\pi} d\phi \int_0^{\infty} dp\;\frac{1}{(p-r\cos{(\theta-\phi)})^2}\;\mathcal{R}f(p,\phi). \tag{4}$$

A second inversion procedure uses circular components $f_l(r)$ (and resp. $\mathcal{R}f_l(p)$) of $f(\mathbf{r})$ (and resp. $\mathcal{R}f(p,\phi)$) [2], defined by

$$f(r,\theta) = \sum_l f_l(r)\, e^{il\theta} \text{ and } \mathcal{R}f(p,\phi) = \sum_l \mathcal{R}f_l(p)\, e^{il\phi}.$$

After substitution in the definition of $\mathcal{R}$ and application of inverse Fourier transform on p for both sides of eq. (3), we find

$$\int_{-\infty}^{+\infty} dp\, e^{2i\pi\xi p}\;\mathcal{R}f_l(p) = \int_0^{\infty} r\, dr\;\, f_l(r)\, 2\pi\, (i^l)\, J_l(2\pi\xi r). \tag{5}$$

Then extraction of $f_l(r)$ in terms of $\mathcal{R}f_l(p)$ can be performed by applying Hankel inverse transform of order l to eq. (5). Hence $f(\mathbf{r})$ can be reconstructed by its circular components.

2.3. **Inversion by back-projection method.** The Fredholm kernel $\delta(p - x\cos\phi - y\sin\phi)$ is the matrix element of $\mathcal{R}$ connecting object space (x,y) to image space (p,ϕ). So if $\mathcal{R}$ acts on functions of object space (x,y), the action of this kernel on functions of image space (p,ϕ) is the adjoint operation $\mathcal{R}^\dagger$.

Definition 3. Let $\mathcal{R}^\dagger f(x,y)$ be the adjoint Radon transform of $\mathcal{R}f(p,\phi)$. It is given by

$$\mathcal{R}^\dagger f(x,y) = \int_{-\infty}^{\infty} dp \int_0^{\pi} d\phi\; \delta(p - x\cos\phi - y\sin\phi)\; \mathcal{R}f(p,\phi). \tag{6}$$

For given ϕ, set $x = x_0 - s\sin\phi$ and $y = y_0 + s\cos\phi$, we observe that the p-integral in (13) is *independent* of s, in fact it is *constant* along the line going through point (x_0,y_0) and of unit normal $\mathbf{n} = (\cos\phi, \sin\phi)$. By this operation, $\mathcal{R}f(p,\theta)$ is said to be *back-projected* along this line. Thus $\mathcal{R}^\dagger_f(x,y)$ is the sum of all back-projected data over all directions ϕ at site (x,y).

Using the identity [2]

$$\int_0^\pi d\phi\ \delta(x\cos\phi + y\sin\phi) = \frac{1}{\sqrt{x^2+y^2}} \tag{7}$$

an explicit expression for $\mathcal{R}^\dagger_f(x,y)$ can be obtained

$$\mathcal{R}^\dagger f(x,y) = \text{p.v.} \int_{\mathbb{R}^2} dx'\,dy'\ \frac{1}{\sqrt{(x-x')^2+(y-y')^2}}\ f(x',y'). \tag{8}$$

This is just a 2D-convolution, hence $f(x,y)$ can be easily extracted in Fourier space in terms of the measured data $\mathcal{R}f(p,\phi)$ and reconstructed in object space.

3. The Conical Radon Transform (CRT)

3.1. Properties.

Definition 4. A "conical projection" of an object described by a density function $f(x,y,z)$ is its integral $\mathcal{C}f(x_S,y_S,\omega)$ on a circular cone of vertical axis direction, vertex (x_S,y_S) on the xOy-plane and opening angle ω:

(9)

$$\int_0^\infty \int_0^\pi r\sin\omega\, d\psi\, dr\, f(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, r\cos\omega),$$

It represents the intensity of scattered radiation coming from point sources on a cone of vertex at (x_S,y_S) and opening angle ω entering the detector along the direction of the cone axis (see figure below).

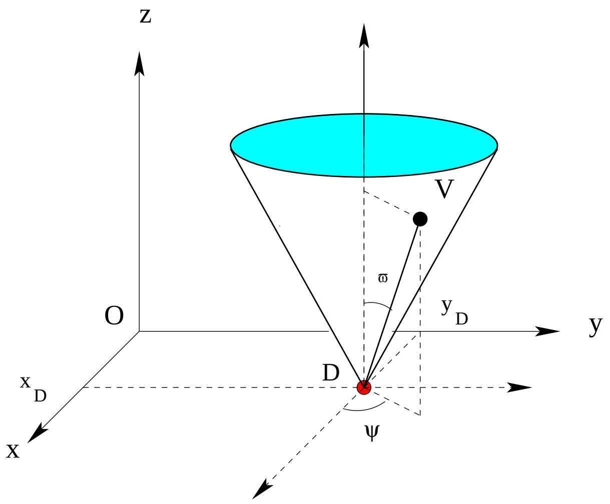
It has a Fredholm integral equation of the first kind form:

$$\mathcal{C}f(x_S,y_S,\omega) = \int_{\mathbb{R}^3} dx\,dy\,dz\ \mathcal{K}(x_S,y_S,\omega|x,y,z)\, f(x,y,z), \tag{10}$$

with $\mathcal{K}(x_S,y_S,\omega|x,y,z) = \delta(\cos\omega\sqrt{(x-x_S)^2+(y-y_S)^2} - z\sin\omega)$.

3.2. Analog of the Central Slice Theorem.

Theorem 5. *Let $\widetilde{f}(p,q,z)$ (resp. $\widetilde{\mathcal{C}f}(p,q,\omega)$) be the Fourier component of $f(x,y,z)$ (resp. of $\widetilde{\mathcal{C}f}(p,q,\omega)$). Then following the*



same procedure as for the Radon transform, we obtain

(11)

$$\widetilde{\mathcal{C}f}(p,q,\omega) = \int_0^\infty r\sin\omega\, dr\, \widetilde{f}(p,q,r\cos\omega)\, 2\pi\, J_0(2\pi\sqrt{p^2+q^2}r\sin\omega).$$

This is of the form of a Hankel transform of order 0. A detailed analysis on the ranges of ω ($0 < \omega < \pi/2$ and $\pi/2 < \omega < \pi$), yields the following inversion formula in 2D-Fourier space (p,q)

$$\widetilde{f}(p,q,z) = (p^2+q^2)\,[Y(z)\int_0^\infty t\,dt\,2\pi\, J_0(2\pi\sqrt{p^2+q^2}zt)\widetilde{\mathcal{C}f}(p,q,t)$$

$$+Y(-z)\int_0^\infty t\,dt\,2\pi\, J_0(2\pi\sqrt{p^2+q^2}zt)\widetilde{\mathcal{C}f}(p,q,-t)], \quad (12)$$

where $Y(z)$ is the Heaviside step function. The full $f(x,y,z)$ is now recovered by inverse 2D-Fourier transform. Note that data is collected for all scattering angle ω, without requiring to move the detector around the object as in a conventional scanning procedure.

3.3. **Back-projection method.** As in the standard Radon case, we construct the adjoint operator $\mathcal{C}^\dagger f(x,yz)$ on functions of image space $\mathcal{C}f(x_S,y_S,\omega)$, by its integral on the δ-function kernel eq. (10)

(13)

$$\int_{\mathbb{R}^2} dx_S\, dy_S \int_0^\pi d\omega\ \delta(\cos\omega\sqrt{(x-x_S)^2+(y-y_S)^2} - z\sin\omega)\ \mathcal{C}f(x_S,y_S,\omega).$$

The ω-integration amounts to integrating on a circle centered at (x,y) and of radius $z\tan\omega$. This yields a function which is *constant* on all lines joining (x, y, z) to $(x_S, y_S, 0)$, hence on a cone of vertex (x, y, z), of vertical axis and with opening angle ω.

By recalling definition 4, we can compute (see [5])

$$\int_{\mathbb{R}^2} dx_S\, dy_S \int_0^{\pi} d\omega \;\; \delta(\cos\omega\sqrt{(x-x_S)^2+(y-y_S)^2} - z\sin\omega)$$

$$\delta(\cos\omega\sqrt{(x'-x_S)^2+(y'-y_S)^2} - z'\sin\omega) = \frac{\sqrt{[(x-x')^2+(y-y')^2]zz'}}{|z^2-z'^2|},$$

to obtain an explicit expression of

$$\mathcal{C}^\dagger f(x,y,z) = \int_{\mathbb{R}^2} dx'\, dy'\, dz' \frac{\sqrt{[(x-x')^2+(y-y')^2]zz'}}{2\pi|z^2-z'^2|}\; f(x',y',z').$$

This is a convolution in the (xOy)-plane and also a convolution in the "*rapidity*"-variable ζ along the Oz-axis, i.e. $z = e^{\zeta}$. So inversion can be achieved conventionally in respective Fourier space, and the final answer obtained by inverse Fourier transform.

4. Conclusion

A new class of Radon transform in $\mathbb{R}^3$, "*imposed*" by the physics of Compton scattering in gamma-ray emission imaging, is presented and new ways to establish its invertibility discussed as altenatives to earlier methods [3, 4, 5]. It serves as foundation for the so-called scattered gamma-ray emission imaging principle, which offers some interesting advantages over existing imaging modalities.

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