Reduced Complexity of a Successive Cancellation Based Decoder for NB-Polar Codes

F. Cochachin, L. Luzzi, and F. Ghaffari
ETIS UMR 8051, CY Cergy Paris Université, ENSEA, CNRS, Cergy, France
{franklin-rafael.cochachin-henostroza, laura.luzzi, fakhreddine.ghaffari}@ensea.fr

***Abstract*—This paper proposes a simplified version of the Successive Cancellation (SC) decoder for Non-Binary Polar (NB-Polar) codes. The proposed decoder, named Successive Cancellation Min-Sum (SC-MS), is exclusively formulated in the Log-Likelihood Ratio (LLR) domain to reduce the decoding complexity of the SC decoder. The NB-Polar codes are associated with Cyclic Code-Shift Keying (CCSK) modulation to obtain a new coded modulation scheme for ultra-low signal-to-noise ratios (SNRs). The quantized version of the SC-MS decoder is investigated using quantized LLRs on optimized size of bits. Our simulation results show that the SC-MS decoder presents a negligible performance degradation with respect to the SC decoder for code length $N < 2048$. Additionally, the 2-bit SC-MS and 3-bit SC-MS offer a good trade-off between decoding performance and complexity.**

I. Introduction

Binary Polar codes [1], introduced by Arikan in 2008, are the first provable capacity-achieving error correction codes for binary-input discrete memoryless channels with low encoding and decoding complexity. Binary polar codes have been adopted for the Enhanced Mobile Broadband (eMBB) control channel of 5G New Radio (NR).

In the current decade, more than 50 billion devices will be connected thanks to the Internet of Things (IoT) [2]. The new generation of mobile networks, notably the 5G system, will need to support short packet traffic [3], [4]. At the link level, we can take advantage of powerful error control codes such as Non-Binary Polar (NB-Polar) codes to transmit short packets. NB-Polar codes [5]–[9] of length $N = 2^n$, also known as q-ary polar codes, are powerful Forward Error Correction (FEC) codes defined over the Galois Field (GF) $\mathbb{F}_q$, $q > 2$. A NB-Polar code processes multiple bits in parallel and helps reduce the probability of frame error compared to binary polar codes. The Cyclic Code-Shift Keying (CCSK) modulation [10] is a q-ary direct-sequence spread-spectrum (DSSS) technique that improves the spectral efficiency of spread-spectrum systems. The CCSK modulation can provide self-synchronization and identification capabilities. All CCSK sequences are obtained from a unique pseudo-random noise (PN) sequence that is circularly shifted. In [11], the authors propose the association of NB-LDPC codes and CCSK to prevent the information loss when calculating channel probabilities at the symbol level.

In this paper, we associate the CCSK modulation to NB-Polar codes to obtain a new coded modulation scheme that can be used in low power networks requiring long range connectivity and high sensitivity, *e.g.* Low-Power Wide-Area (LPWA) networks for IoT applications. We examine the Successive Cancellation (SC) decoder for NB-Polar codes operating in the probability domain. The association of CCSK and NB-Polar codes enables the SC decoder to have good decoding performance at ultra-low signal-to-noise ratios (SNRs). In order to reduce the complexity of the SC decoder, we propose a decoder named Successive Cancellation Min-Sum (SC-MS) decoder that is exclusively formulated in the Log-Likelihood Ratio (LLR) domain. We investigate SC-MS decoders defined over small alphabets constructed from $Q_m \in \{2, 3, 4, 5\}$ bits of precision for the *internal LLRs*. We also quantize the *channel LLRs* on small alphabets constructed from $Q_{ch} \in \{2, 3, 4, 5\}$ bits of precision.

Our numerical results show that the SC-MS decoder presents a negligible performance degradation with respect to the SC decoder for code length $N \in \{64, 128, 256, 512, 1024\}$. In the case of code length $N = 2048$, the SC-MS decoder suffers a very small performance loss compared to the SC decoder, the degradation at FER $= 10^{-2}$ is around 0.1 dB for $\mathbb{F}_{2^8}$. Comparing quantized SC-MS decoders, we observe that the $(Q_{ch} = 3, Q_m = 4)$-bit SC-MS can achieve almost the same performance as the $(Q_{ch} = 5, Q_m = 5)$-bit SC-MS. We also observe that the $(Q_{ch} = 2, Q_m = 3)$-bit SC-MS decoder offers a good trade-off between performance and complexity.

The outline of the paper is as follows. Section II presents NB-Polar codes and the CCSK modulation. In Section III, we describe the LLR calculation of the association between the CCSK modulation and the NB-Polar code, and we propose the SC-MS decoder. Section IV presents the finite precision SC-MS decoder. Simulation results are presented in Section V. Finally, Section VI concludes the paper.

II. NB-Polar Codes and CCSK Modulation

A. Non-Binary Polar Codes

In this paper, a NB-Polar code of length $N = 2^n$ is defined over the Galois Field $\mathbb{F}_q$, with $q = 2^p$ and where $p > 1$. Let $\boldsymbol{x} = (x_0, ..., x_{N-1})$, $x_i \in \mathbb{F}_q$ for $i = 0, ..., N-1$, be the codeword that is obtained after encoding the message $\boldsymbol{u} = (u_0, ..., u_{N-1})$, $u_i \in \mathbb{F}_q$ for $i = 0, ..., N-1$. Also, let $\mathbb{F}_q^*$ denote the set of all non-zero elements of $\mathbb{F}_q$. It is worth mentioning that a GF element can be represented by a binary vector of size p, *e.g.* $x_1 = (x_1^0, ..., x_1^{p-1})$. For the construction of NB-Polar codes, we consider the kernel transformation $\mathcal{T} : (x_0, x_1) = (u_0 \oplus u_1, h \odot u_1)$, with '$\oplus$' and '$\odot$' respectively denoting the addition and multiplication rules over $\mathbb{F}_q$, and

978-1-6654-0943-8/21/$31.00 ©2021 IEEE

978-1-6654-0943-8/21/$31.00 ©2021 IEEE

where $h \in \mathbb{F}_q^*$ is a randomly chosen element. It has been shown in [5] that the transformation $\mathcal{T}$ guarantees polarization of the NB-Polar codes if q is a prime power. A NB-Polar code of length $N = 2^n$ is obtained by applying $\mathcal{T}$ recursively (n times), and in each step of the recursion different values of h can be used. In the decoding process, $\mathcal{T}$ is composed of a check node (CN) and a variable node (VN), hence a NB-Polar code of $N = 2^n$ has $n(N/2)$ VNs/CNs. The update rule for a VN/CN is presented in section III. Fig. 1 shows a graph of a NB-Polar code of $N = 2^3$ and composed of 12 VNs (represented with $\boxed{=}$) and 12 CNs (represented with $\bigoplus$).

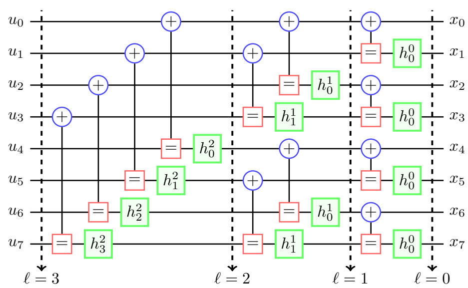


Fig. 1: Graph of a NB-Polar code of length $N = 8$.

A (N, K) NB-Polar code has K information symbols, a total length of $N = 2^n$, $N - K$ frozen symbols, and code rate $R_c = K/N$. Thus a codeword is composed of $K \times p$ bits of information. Let $\mathcal{I} = \{i_0, ..., i_{K-1}\}$ denote the set of indices $0 \leq i_K \leq N-1$ that serves to indicate the positions of the information symbols, and let $\mathcal{I}^c$ denote the set of all elements in the set $\{0, ..., N-1\}$ that are not in $\mathcal{I}$. We use a genie-aided successive cancellation decoder [1] to compute the sets $\mathcal{I}$ and $\mathcal{I}^c$ at each SNR value. In this paper, the frozen symbols $\boldsymbol{u}_{\mathcal{I}^c} = \{u_i, i \in \mathcal{I}^c\}$ are set to zero and their positions are known to both encoder and decoder. Note that $\boldsymbol{u} = (\boldsymbol{u}_{\mathcal{I}}, \boldsymbol{u}_{\mathcal{I}^c})$, where $\boldsymbol{u}_{\mathcal{I}} = \{u_i, i \in \mathcal{I}\}$ are the information symbols.

B. Cyclic Code-Shift Keying Modulation

The CCSK modulation is a technique that associates pseudo-random noise (PN) sequences of length $q = 2^p$ to p-bit symbols. Each sequence is derived from a unique PN sequence $\eta_0 = (\eta_0(0), ..., \eta_0(q-1))$ of length q, where $\eta_0(k) \in \{0, 1\}$ for $k = 0, ..., q-1$. We perform the circular shift to the left of η_0 in $s \in \{0, ..., q-1\}$ positions to obtain the PN sequence $\eta_s = (\eta_s(0), ..., \eta_s(q-1))$, where $\eta_s(k) = \eta_0((k+s) \mod q)$ for $k = 0, ..., q-1$. The rate of the modulation is $R_m = p/q$. PN sequences can be constructed using a Linear Feedback Shift Register (LFSR). The PN sequence generated by an LFSR has good autocorrelation properties. Algorithms like the genetic algorithm can also be used to generate and optimize a PN sequence.

III. Successive Cancellation Based Decoders

In this section, we present the association of CCSK modulation to NB-Polar codes. We also propose a simplified version of the SC decoder. Taking advantage of the recursive structure, the decoder for the NB-Polar code proceeds in $n+1$ stages $\ell \in \{0, ...n\}$, see Fig 1. From this section onwards, we distinguish between two types of LLRs, the *channel LLRs* calculated at stage $\ell = 0$ and the *internal LLRs* obtained at stage $\ell \in \{1, ..., n\}$.

A. Definition of the Log-Likelihood Ratio

Let $\eta_0 = (\eta_0(0), ..., \eta_0(q-1))$ denote a fundamental PN sequence, and let $\boldsymbol{\eta} = (\eta_{x_0}, ..., \eta_{x_{N-1}})$ denote the codeword after applying the CCSK technique to $\boldsymbol{x} = (x_0, ..., x_{N-1})$. Each symbol $x_i \in \mathbb{F}_q$, $i = 0, ..., N-1$, of $\boldsymbol{x}$ is mapped to the PN sequence $\eta_{x_i} = (\eta_{x_i}(0), ..., \eta_{x_i}(q-1))$, where $\eta_{x_i}(k) = \eta_0((k + x_i) \mod q)$ for $k = 0, ..., q-1$, with $\eta_{x_i}(k) \in \{0, 1\}$. The code rate after the CCSK modulation is $R_e = Rc \times R_m = (K \times p)/(N \times q)$. We consider that $\boldsymbol{\eta}$ is modulated by the Binary Phase-Shift Keying (BPSK) modulation and transmitted over the Binary Input Additive White Gaussian Noise (BI-AWGN) channel with noise variance σ^2. The channel output $\boldsymbol{y} = (y_0, ..., y_{N-1})$, with $y_i = (y_i(0), ..., y_i(q-1))$ for $i = 0, ..., N-1$, is modeled by $y_i(k) = (1 - 2\eta_{x_i}(k)) + z_i(k)$, $k = 0, \ldots, q-1$, where $z_i(k)$ is a sequence of independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance σ^2. We can define the vector of channel LLRs $L_i = (L_i(0), ..., L_i(q-1))$ of a GF symbol x_i as

$$L_i(x_i) = \log\left(\frac{\Pr(y_i \mid \hat{\eta}_{x_i})}{\Pr(y_i \mid \eta_{x_i})}\right) \quad \forall x_i \in \mathbb{F}_q, \tag{1}$$

where $\hat{\eta}_{x_i}$ is the hard decision over y_i, *i.e.* $\hat{\eta}_{x_i}(k) = 1$ if $y_i(k) < 0$, $\hat{\eta}_{x_i}(k) = 0$ otherwise, $k = 0, \ldots, q-1$. Equation (1) can be expressed as:

$$L_i(x_i) = \sum_{k=0}^{q-1} \log\left(\Pr(y_i(k) \mid \hat{\eta}_{x_i}(k))\right) - \log\left(\Pr(y_i(k) \mid \eta_{x_i}(k))\right) \ \forall x_i \in \mathbb{F}_q. \tag{2}$$

Replacing the conditional distribution for the BI-AWGN channel $\Pr(y \mid x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x)^2/2\sigma^2}$, we obtain

$$L_i(x_i) = \sum_{k=0}^{q-1} \frac{2y_i(k)}{\sigma^2} \left(\eta_{x_i}(k) - \hat{\eta}_{x_i}(k)\right) \quad \forall x_i \in \mathbb{F}_q. \tag{3}$$

The vector of channel LLRs computed with (3) allows us to obtain only positive LLR values. To set at least one element of L_i equal to zero, we use

$$L_i'(x_i) = L_i(x_i) - \min(L_i) \quad \forall x_i \in \mathbb{F}_q. \tag{4}$$

From the LLR values, the probability distribution $P_i(x_i) = \Pr(y_i \mid x_i)$ can be computed by:

$$P_i(x_i) = \frac{e^{-L_i(x_i)}}{\sum\limits_{x_i' \in \mathbb{F}_q} e^{-L_i(x_i')}} \quad \forall x_i \in \mathbb{F}_q. \tag{5}$$

B. Successive Cancellation Decoder

The SC decoder, defined in the probability domain [12], estimates the information symbols one by one. Note that each estimated symbol is used to decode the next symbol.

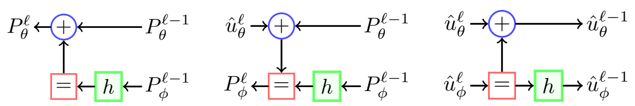
Let $\boldsymbol{P}^\ell = (P_0^\ell, ..., P_{N-1}^\ell)$ denote the probability distribution computed at stage $\ell \in \{0, ..., n\}$ during the decoding process, where $P_i^\ell = (P_i^\ell(0), ..., P_i^\ell(q-1))$, $i = 0, ..., N-1$, denotes the probability distribution of the intermediate values u_i^ℓ, and let $\hat{u}_i^\ell$ denote the estimated symbol of u_i^ℓ. We note that $u_i^0 = x_i$. At the initialization stage of the SC decoder ($\ell = 0$), P_i^0 is initialized with (5), *i.e.* $P_i^0 = P_i(x_i)$ for $i = 0, ..., N-1$.
Let us consider the transformation

$$\mathcal{T} : \left(u_{\theta_t^{\ell-1}}^{\ell-1}, u_{\phi_t^{\ell-1}}^{\ell-1}\right) = \left(u_{\theta_t^{\ell-1}}^{\ell} \oplus u_{\phi_t^{\ell-1}}^{\ell}, h_{\phi_t^{\ell-1}}^{\ell-1} \odot u_{\phi_t^{\ell-1}}^{\ell}\right), \quad (6)$$

where $\theta_t^{\ell-1}$ and $\phi_t^{\ell-1}$ are given by

$$\theta_t^{\ell-1} = 2t - (t \mod 2^{\ell-1}),$$
$$\phi_t^{\ell-1} = 2^{\ell-1} + 2t - (t \mod 2^{\ell-1}),$$

for $t = 0, 1, ..., N/2 - 1$. To simplify the notations, we use θ (respectively ϕ) to denote $\theta_t^{\ell-1}$ (respectively $\phi_t^{\ell-1}$). Similarly, h is used to denote $h_{\phi_t^{\ell-1}}^{\ell-1}$ for simplicity.
The update rules considering $\mathcal{T}$ are shown in Fig. 2.



(a) Check node. (b) Variable node. (c) Propagation of $\hat{u}$.

Fig. 2: Update rules for the SC decoder.

The update rule at a CN is given by

$$P_\theta^\ell(u_\theta^\ell) = \beta \sum_{u_\phi^\ell \in \mathbb{F}_q} P_\theta^{\ell-1}(u_\theta^\ell \oplus u_\phi^\ell) P_\phi^{\ell-1}(h \odot u_\phi^\ell) \quad \forall u_\theta^\ell \in \mathbb{F}_q, \quad (7)$$

where $\beta^{-1} = \sum_{u_\theta^\ell \in \mathbb{F}_q} P_\theta^\ell\left(u_\theta^\ell\right)$, *i.e.* β is a normalization factor.
The update rule at a VN is defined as

$$P_\phi^\ell\left(u_\phi^\ell\right) = \delta P_\theta^{\ell-1}\left(\hat{u}_\theta^\ell \oplus u_\phi^\ell\right) P_\phi^{\ell-1}\left(h \odot u_\phi^\ell\right), \quad (8)$$

where $\delta^{-1} = \sum_{u_\phi^\ell \in \mathbb{F}_q} P_\phi^\ell\left(u_\phi^\ell\right)$.
To propagate the estimated symbols, we use

$$\left(\hat{u}_\theta^{\ell-1}, \hat{u}_\phi^{\ell-1}\right) = \left(\hat{u}_\theta^\ell \oplus \hat{u}_\phi^\ell, h \odot \hat{u}_\phi^\ell\right). \quad (9)$$

We can estimate a message $\hat{\boldsymbol{u}} = (\hat{u}_0^n, ..., \hat{u}_{N-1}^n)$ of length $N = 2^n$ by applying (7), (8), and (9) at each decoding stage ℓ. At stage $\ell = n$, the hard decision of u_i^n is obtained as

$$\hat{u}_i^n = \begin{cases} 0, & \text{if } \quad i \in \mathcal{I}^c \\ \arg\max_{u_i^n} P_i^n(u_i^n), & \text{if } \quad i \in \mathcal{I}. \end{cases} \quad (10)$$

C. Successive Cancellation Min-Sum Decoder

We propose a simplified version of the SC decoder named Successive-Cancellation Min-Sum (SC-MS) decoder that is exclusively formulated in the LLR domain.
Let $\boldsymbol{L}^\ell = (L_0^\ell, ..., L_{N-1}^\ell)$ denote the LLRs computed at stage $\ell \in \{0, ..., n\}$ during the decoding process, and let $L_i^\ell = (L_i^\ell(0), ..., L_i^\ell(q-1))$, $i = 0, ..., N-1$, denote the LLR of u_i^ℓ. The SC-MS decoder is initialized using (4), *i.e.* we have $L_i^0 = L_i'(x_i)$ for $i = 0, ..., N-1$. With these notations and considering $\mathcal{T}$ of (6), the update rule at a CN is given by

$$L_\theta^\ell(u_\theta^\ell) = \min_{u_\phi^\ell \in \mathbb{F}_q} \left(L_\theta^{\ell-1}(u_\theta^\ell \oplus u_\phi^\ell) + L_\phi^{\ell-1}(h \odot u_\phi^\ell)\right) \quad \forall u_\theta^\ell \in \mathbb{F}_q. \quad (11)$$

The update rule at a VN is defined as

$$L_\phi'\left(u_\phi^\ell\right) = L_\theta^{\ell-1}\left(\hat{u}_\theta^\ell \oplus u_\phi^\ell\right) + L_\phi^{\ell-1}\left(h \odot u_\phi^\ell\right). \quad (12)$$

$$L_\phi^\ell\left(u_\phi^\ell\right) = L_\phi'\left(u_\phi^\ell\right) - \min\left(L_\phi'\right). \quad (13)$$

Equation (13) is necessary for numerical reasons to ensure the nondivergence of the SC-MS decoder. In the decoding process without the use of (13), the internal LLRs converge to very large numerical values that are computationally intractable.
We can estimate $\hat{\boldsymbol{u}} = (\hat{u}_0^n, ..., \hat{u}_{N-1}^n)$ by applying (11), (12), (13), and (9), where the hard decision of u_i^n is obtained as

$$\hat{u}_i^n = \begin{cases} 0, & \text{if } \quad i \in \mathcal{I}^c \\ \arg\min_{u_i^n} L_i^n(u_i^n), & \text{if } \quad i \in \mathcal{I}. \end{cases} \quad (14)$$

D. Complexity of SC-based Decoders

We measure decoding complexity by counting the adders, multipliers, and comparators used by the decoder over the field $\mathbb{F}_q$. In an SC decoder, a CN requires q^2 multiplications and $q(q-1)$ additions; and a VN requires q multiplications. The normalization process requires $q-1$ additions and q divisions. In the case of an SC-MS decoder, a CN requires q^2 additions and $q(q-1)$ comparisons; and a VN requires q additions, $q-1$ comparisons, and q subtractions. We report in Table I the complexity of a CN and a VN. Note that the subtractions are considered as additions and the divisions as multiplications.

TABLE I: Complexity of a single CN and a single VN for $\mathbb{F}_q$.

Decoder	Node	# of multipliers	# of adders	# of comparators	Area† (mm^2)	Power† (W)
SC	CN	q^2+q	q^2-1	-	344.91	37.745
	VN	$2q$	$q-1$	-	8.18	1.026
SC-MS	CN	-	q^2	$q(q-1)$	169.11	9.147
	VN	-	$2q$	$q-1$	5.11	0.281

† The area utilization and power consumption for the field $\mathbb{F}_{2^6}$.

To make a rough estimation of complexity, we consider floating point calculations in single precision format with 0.18 μm technology. Based on results presented in [13], a simple precision addition requires an area of 38560 μm^2 and consumes 2.15 mW, and a multiplication occupies an area of 44953 μm^2 and consumes 6.957 mW. In [14], a comparator has an area of 2771 μm^2 and consumes 84.52 μW. Considering the field $\mathbb{F}_{2^6}$ and assuming that all operations are carried out in parallel, the kernel node (a VN + a CN) of the SC-MS reduces the area by 50.66% (from 353.09 mm^2 to 174.22 mm^2), also one can see a power consumption saving of 75.68% (from 38.771 W to 9.428 W). We can clearly see that the SC-MS decoder reduces hardware complexity and helps save power.

IV. Finite Precision SC-MS Decoders

In this section, we present a finite precision version of the SC-MS decoder for low complexity hardware implementation.

A. Quantization used for SC-MS Decoders

For quantized SC-MS decoders, the LLRs have to be quantized and saturated. Let Q_{ch} denote the number of precision bits of quantized channel LLRs, and let $\mathcal{A}_{ch}$ denote the alphabet of channel LLRs defined as $\mathcal{A}_{ch} = \{0, +1, ..., +N_{ch}\}$, composed of $N_{ch} + 1$ states, where $N_{ch} = 2^{Q_{ch}} - 1$.
Let us also denote the quantizer by $\mathcal{Q} : \mathbb{R} \rightarrow \mathcal{A}_{ch}$ for the quantized SC-MS decoders, defined as

$$\mathcal{Q}(a) = \min\left(\lfloor \alpha \times a \rfloor, N_{ch}\right), \tag{15}$$

where $\lfloor . \rfloor$ depicts the floor function. The parameter α is called *channel gain factor* and is used to increase or decrease the amplitude of channel LLRs at the decoder input.
After computing L_i' with (3) and (4), the quantized version $I_i = (I_i(0), ..., I_i(q-1))$ of L_i' can be obtained as

$$I_i(x_i) = \mathcal{Q}(L_i'(x_i)) \quad \forall x_i \in \mathbb{F}_q. \tag{16}$$

The quantized channel LLR I_i is used to initialize the decoder. Let us denote by $\mathcal{A}_L$ the alphabet of the internal LLRs defined as $\mathcal{A}_L = \{0, +1, ..., +N_m\}$, composed of $N_m + 1$ states, with $N_m = 2^{Q_m} - 1$ and where Q_m is the number of precision bits of quantized internal LLRs.

B. Update Rules for SC-MS Decoders

Let $\boldsymbol{I}^\ell = (I_0^\ell, ..., I_{N-1}^\ell)$ denote the quantized LLRs computed at stage $\ell \in \{0, ..., n\}$ during the decoding process, and let $I_i^\ell = (I_i^\ell(0), ..., I_i^\ell(q-1))$, $i = 0, ..., N-1$, denote the quantized LLR of u_i^ℓ. At stage $\ell = 0$, the input of the quantized decoder is obtained with (16), *i.e.* $I_i^0 = I_i(x_i)$ for $i = 0, ..., N-1$.
Considering $\mathcal{T}$ of (6), the update rule at a CN is given by

$$I_\theta^\ell\left(u_\theta^\ell\right) = \min_{u_\phi^\ell \in \mathbb{F}_q}\left(I_\theta^{\ell-1}\left(u_\theta^\ell \oplus u_\phi^\ell\right) + I_\phi^{\ell-1}\left(h \odot u_\phi^\ell\right)\right) \; \forall u_\theta^\ell \in \mathbb{F}_q. \tag{17}$$

For the case of a VN, the update rule is determined by

$$I_\phi'\left(u_\phi^\ell\right) = I_\theta^{\ell-1}\left(\hat{u}_\theta^\ell \oplus u_\phi^\ell\right) + I_\phi^{\ell-1}\left(h \odot u_\phi^\ell\right). \tag{18}$$

$$I_\phi^\ell\left(u_\phi^\ell\right) = \min\left(I_\phi'\left(u_\phi^\ell\right) - \min\left(I_\phi'\right), N_m\right). \tag{19}$$

The message $\hat{\boldsymbol{u}} = (\hat{u}_0^n, ..., \hat{u}_{N-1}^n)$ can be estimated using (17), (18), (19), and (9). The hard decision u_i^n is obtained using (14), where L_i^n is replaced by I_i^n for $i = 0, ..., N-1$.
The operators used in the quantized SC-MS decoder require $Q_m + 1$ bits of precision. Since it is easier to implement fixed point operations, the (Q_{ch}, Q_m)-bit SC-MS decoder reduces computational complexity. Comparing the $(64, 64)$-bit SC-MS (floating-point) with the $(2, 3)$-bit SC-MS, we can roughly obtain a 93.7% reduction in complexity. Note that we compare the number of bits of the operators, *i.e.* $100(1 - (Q_m + 1)/64)$.

V. Numerical Results

In this section, we present the frame error rate (FER) performance of SC-based decoders over the BI-AWGN channel for various NB-Polar codes and two high-order GFs. A genie-aided SC decoder is used to compute the set $\mathcal{I}$. Additionally, the set of symbol positions $\mathcal{I}_{MS}$ is obtained using a genie-aided SC-MS decoder. The sets $\mathcal{I}$ and $\mathcal{I}_{MS}$ are calculated at each value of SNR; the results show that the two sets are not equal although they may have many elements in common.

A. Performance of Floating-Point SC-based Decoders

We present the Monte Carlo simulations for (N, K) NB-Polar codes with CCSK modulation at very low SNR considering four cases: *(i)* $(K = 20, q = 64)$, *(ii)* $(K = 160, q = 64)$, *(iii)* $(K = 30, q = 256)$, and *(iv)* $(K = 240, q = 256)$. We consider $N \in \{64, 128, 256\}$ for $K \in \{20, 30\}$, and $N \in \{512, 1024, 2048\}$ for $K \in \{160, 240\}$. For all NB-Polar codes, the SC decoder performance is shown as a benchmark. Fig. 3 shows the FER performance comparison between the SC and SC-MS using $\mathcal{I}$ and $\mathcal{I}_{MS}$, for three code lengths $N \in \{64, 128, 256\}$ with $K = 20$, and for the GF $\mathbb{F}_{2^6}$. The results obtained show that the SC-MS decoders present a negligible performance degradation with respect to the SC decoders. Considering the case of $K = 30$ and the field $\mathbb{F}_{2^8}$, the simulation results also show a negligible performance loss for SC-MS decoders when the code length is $N \in \{64, 128, 256\}$.

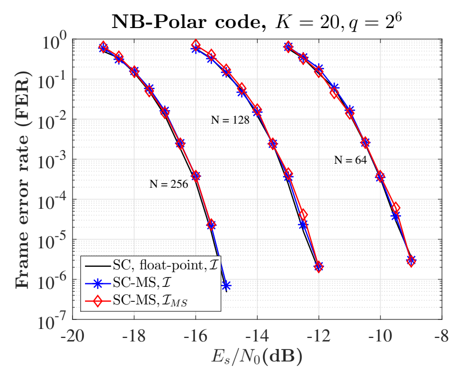


Fig. 3: FER performance of SC-based decoders for $K = 20$ and $N \in \{64, 128, 256\}$ over $\mathbb{F}_{2^6}$.

Simulation results for the code length $N \in \{512, 1024, 2048\}$ and the field $\mathbb{F}_{2^8}$ are provided in Fig. 4. We observe at FER $= 10^{-2}$ a very small performance loss of about 0.1 dB for $(N = 2048, K = 240)$, 0.08 dB for $(N = 1024, K = 240)$, and 0.05 dB for $(N = 512, K = 240)$. Similar behavior is observed for $K = 160$, $\mathbb{F}_{2^6}$, and $N \in \{512, 1024, 2048\}$.
When comparing the FER performance of the SC-MS decoders using the sets $\mathcal{I}$ and $\mathcal{I}_{MS}$, the SC-MS decoders implemented with $\mathcal{I}$ have (almost) the same decoding performance as SC-MS decoders implemented with $\mathcal{I}_{MS}$.

B. Performance of Fixed-Point SC-MS Decoders

In this section, we present the simulation results of finite precision SC decoders over the field $\mathbb{F}_{2^6}$. We focus on the $(N = 64, K = 20)$ NB-Polar code and different bits of precision for channel LLRs and internal LLRs. All quantized decoders use the set $\mathcal{I}$ for the symbol positions.
The channel gain factor α used to quantize the channel LLRs are optimized using Monte Carlo simulations. In Table II, we indicate the optimal values of α for (Q_{ch}, Q_m)-bit SC-MS

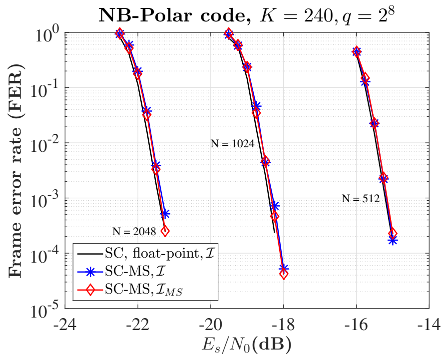


Fig. 4: FER performance of SC-based decoders for $K = 240$ and $N \in \{512, 1024, 2048\}$ over $\mathbb{F}_{2^8}$.

decoders. We also report in Table II the SNR losses of the (Q_{ch}, Q_m)-bit SC-MS decoder with respect to the SC decoder.

TABLE II: SNR losses of finite precision SC-MS decoders for the $(N = 64, K = 20)$ NB-Polar code and for the field $\mathbb{F}_{2^6}$.

(Q_{ch}, Q_m)	α^*	SNR loss (dB) @ FER = 10^{-2}	(Q_{ch}, Q_m)	α^*	SNR loss (dB) @ FER = 10^{-2}
$(2, 2)$	0.40	1.00	$(3, 4)$	0.90	0.17
$(2, 3)$	0.55	0.45	$(3, 5)$	0.90	0.15
$(2, 4)$	0.55	0.45	$(4, 4)$	1.10	0.14
$(2, 5)$	0.55	0.45	$(4, 5)$	1.40	0.11
$(3, 3)$	0.60	0.38	$(5, 5)$	1.90	0.07

Fig. 5 depicts the FER performance of the best quantized SC-MS decoders for low precision (Q_{ch}, Q_m). We can see that the $(5, 5)$-bit SC-MS decoder has the best FER performance, and the $(3, 4)$-bit SC-MS decoder can achieve almost the same performance as the $(5, 5)$-bit SC-MS decoder. From all these results, the $(2, 3)$-bit SC-MS decoder is a good option for applications that require low hardware complexity. In the case where the decoding performance is privileged by the applications, the $(3, 4)$-bit SC-MS decoder is a good choice for a hardware implementation.

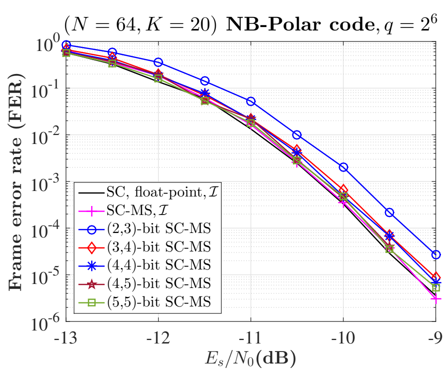


Fig. 5: FER performance of the (Q_{ch}, Q_m)-bit SC-MS decoders over $\mathbb{F}_{2^6}$, $N = 64$, and $K = 20$.

VI. Conclusion

In this paper, we have shown that NB-Polar codes associated with CCSK modulation can achieve very low FER at ultra-low SNR. Moreover, we have proposed a simplified version of the SC decoder. The SC-MS decoder has been defined in the LLR domain and it reduces the decoding complexity. We have also proposed a quantized version of the SC-MS decoder.
The Monte Carlo simulations have shown that the SC-MS decoders present a negligible performance degradation with respect to the SC decoders for code length $N \in \{64, 128, 256, 512, 1024\}$. In this study, we have demonstrated that the $(2, 3)$-bit SC-MS and $(3, 4)$-bit SC-MS offer a good trade-off between decoding performance and complexity.

Acknowledgement

This work has been funded by the french ANR under grant number ANR-19-CE25-0013-01 (QCSP Project: https://qcsp.univ-ubs.fr/).
The authors would like to thank Valentin Savin and Emmanuel Boutillon for their helpful discussions.

References

[1] E. Arikan, "Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels," *IEEE Transactions on Information Theory*, vol. 55, no. 7, pp. 3051–3073, 2009.
[2] K. Mekki, E. Bajic, F. Chaxel, and F. Meyer, "A Comparative Study of LPWAN Technologies for Large-Scale IoT Deployment," *ICT Express*, vol. 5, no. 1, pp. 1–7, 2019.
[3] G. Durisi, T. Koch, and P. Popovski, "Toward Massive, Ultrareliable, and Low-Latency Wireless Communication With Short Packets," *Proceedings of the IEEE*, vol. 104, no. 9, pp. 1711–1726, 2016.
[4] A. Bana, K. F. Trillingsgaard, P. Popovski, and E. de Carvalho, "Short Packet Structure for Ultra-Reliable Machine-Type Communication: Tradeoff Between Detection and Decoding," in *2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 6608–6612, 2018.
[5] E. Şaşoğlu, E. Telatar, and E. Arikan, "Polarization for Arbitrary Discrete Memoryless Channels," in *2009 IEEE Information Theory Workshop*, pp. 144–148, 2009.
[6] R. Mori and T. Tanaka, "Channel Polarization on q-ary Discrete Memoryless Channels by Arbitrary Kernels," in *2010 IEEE International Symposium on Information Theory*, pp. 894–898, 2010.
[7] E. Şaşoğlu, "Polar Codes for Discrete Alphabets," in *2012 IEEE International Symposium on Information Theory Proceedings*, pp. 2137–2141, 2012.
[8] W. Park and A. Barg, "Polar Codes for Q-Ary Channels, $q = 2^r$," *IEEE Transactions on Information Theory*, vol. 59, no. 2, pp. 955–969, 2013.
[9] M. Chiu, "Non-Binary Polar Codes with Channel Symbol Permutations," in *2014 International Symposium on Information Theory and its Applications*, pp. 433–437, 2014.
[10] A. Y.-C. Wong and V. C. M. Leung, "Code-Phase-Shift Keying: A Power and Bandwidth Efficient Spread Spectrum Signaling Technique for Wireless Local Area Network Applications," in *CCECE '97. Canadian Conference on Electrical and Computer Engineering. Engineering Innovation: Voyage of Discovery. Conference Proceedings*, vol. 2, pp. 478–481 vol.2, 1997.
[11] O. Abassi, L. Conde-Canencia, M. Mansour, and E. Boutillon, "Non-Binary Low-Density Parity-Check Coded Cyclic Code-Shift Keying," in *2013 IEEE Wireless Communications and Networking Conference (WCNC)*, pp. 3890–3894, 2013.
[12] L. Chandesris, *Contribution à la Construction et au Décodage des Codes Polaires*. PhD thesis, Université de Cergy-Pontoise, Cergy-Pontoise, France, 2019.
[13] B. N. Reddy and P. A. S. Beulet, "An Efficient Multi-Precision Floating Point Adder and Multiplier," *Indian Journal of Science and Technology*, vol. 8, no. 25, pp. 1–5, 2015.
[14] A. T. Samuel and J. Senthilkumar, "A novel floating point comparator using parallel tree structure," in *2015 International Conference on Circuits, Power and Computing Technologies [ICCPCT-2015]*, pp. 1–6, 2015.